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Non-equivalent n-norms being Sequentially Equivalent

By

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Abstract

In [4], we have already studied the space of p-summable sequences i.e. (l^p) as an n-normed space by defining a new n-norm on it. In [5], we have resulted that equivalent norms can be derived by non-equivalent n-norms. Inspired by the problem raised in Gunawan and others paper [7], in this paper, we shall show that sequentially equivalent n-norms need not be equivalent.

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1. Introduction

Gähler [8], initialy introduced the theory of *2-norm* on a linear space while that of *n-norm* can be found in [10] and has been studied in many papers such as [1, 9, 3]. Research works on sequence spaces regarded as *n-normed space* can be found in [1, 2, 4, 5, 6].

Definition 1.1. Let $\mathbf{X}$ be a vector space over $\mathbb{K}(=\mathbb{R}$ or $\mathbb{C})$ of dimension $d \geq n (n \geq 2)$. A non-negative real valued function $\|.,\ldots,.\|$ defined on $\mathbf{X^n}$ satisfying the four conditions:

(N1) $\|x^1, x^2, \cdots, x^n\| = 0$ if and only if $x^1, x^2, \cdots, x^n$ are linearly dependent;
(N2) $\|x^1, x^2, \cdots, x^n\|$ is invariant under the permutation of $x^1, x^2, \cdots, x^n$;
(N3) $\|\alpha \cdot x^1, x^2, \cdots, x^n\| = |\alpha| \cdot \|x^1, x^2, \cdots, x^n\|$;
(N4) $\|x^1 + y, x^2, \cdots, x^n\| \leq \|x^1, x^2, \cdots, x^n\| + \|y, x^2, \cdots, x^n\|$;

for all $x^1, x^2, \cdots, x^n, y \in \mathbf{X}$ and for all $\alpha \in \mathbb{K}$, is called an **n-norm on X**, and the pair $(\mathbf{X}, \|.,\ldots,.\|)$ is called an **n-normed space**.

Definition 1.2. A sequence $\left(x^l\right)_{l=0}^{\infty}$ defined in n-normed space $(\mathbf{X}, \|.,\ldots,.\|)$ is

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said to be **convergent** at $x \in \mathbf{X}$ if

$$\|x^l - x, z^1, \cdots, z^{n-1}\| \to 0 \qquad \text{as} \qquad l \to \infty$$

for every $z^1, \cdots, z^{n-1} \in \mathbf{X}$.

Definition 1.3. Two n-norms $\|\cdot, \cdots, \cdot\|_1$ and $\|\cdot, \cdots, \cdot\|_2$ defined on a linear space $\mathbf{X}$ are said to be **equivalent** or **equivalent of type 1** (in short **E1**) if $\exists\, K_1, K_2 > 0$ such that:

$$K_1 \cdot \|x^1, x^2, \cdots, x^n\|_1 \leq \|x^1, x^2, \cdots, x^n\|_2 \leq K_2 \cdot \|x^1, x^2, \cdots, x^n\|_1$$

for all $x^1, x^2, \cdots, x^n \in \mathbf{X}$. n-norms which are not equivalent are termed as non-equivalent.

Definition 1.4. Two n-norms $\|\cdot, \cdots, \cdot\|_1$ and $\|\cdot, \cdots, \cdot\|_2$ defined on a linear space $\mathbf{X}$ are said to be **sequentially equivalent** or **Sequentially equivalent of type 1** (in short **SE1**) if convergence of a sequence in $\|\cdot, \cdots, \cdot\|_1$ implies convergence in $\|\cdot, \cdots, \cdot\|_2$ and vice versa.

Definition 1.5. Let $(\mathbf{X}, \|\cdot, \cdots, \cdot\|)$ be an n-normed space and $\{e^1, \cdots, e^n\}$ is a set of linearly independent vectors in $\mathbf{X}$ then both of the functions $\|\cdot\|_\infty^d$ and $\|\cdot\|_q^d$ define a norm on $\mathbf{X}$ (known as **derived norm** with respect to the set $\{e^1, \cdots, e^n\}$) and they are equivalent, where

(1) $\|x\|_\infty^d = max\left\{\|x, e^{t_1}, \ldots, e^{t_{n-1}}\| : \{t_1, \ldots, t_{n-1}\} \subset \{1, \ldots, n\}\right\}$
(2) $\|x\|_q^d = \left(\sum_{\{t_1,\ldots,t_{n-1}\}\subset\{1,\ldots,n\}} \|x, e^{t_1}, \ldots, e^{t_{n-1}}\|^q\right)^{1/q}; \quad 1 \leq q < \infty.$

It is obvious that if a sequence is convergent in an n-normed space then it is convergent in its derived normed space also.

In this paper, we shall study the sequence space l^p where

$$l^p = \left\{ x = (x_i)_{i=0}^\infty : \sum_{i=0}^\infty |x_i|^p < \infty \quad \text{where} \quad x_i \in \mathbb{K}, \quad \text{for all} \quad i = 0, 1, 2, \ldots \right\}.$$

As we know that $(l^p, \|\cdot\|_p)$ is a Banach space where $\|x\|_p = (\sum_{i=0}^\infty |x_i|^p)^{1/p}$ while $(l^p, \|\cdot\|_\infty)$ forms simply a normed space where $\|x\|_\infty = \sup_{0\leq i<\infty} |x_i|$.

In [4], for our convenience and need we have denoted the set of whole numbers as $\mathbb{N} = \{0, 1, 2, \ldots\}$, which is also considered as a sequence $\mathbb{N} = (0, 1, 2, \ldots)$. Further, we have denoted the sequence $\mathbb{N} = (0, 1, 2, \ldots)$ in the form of *n-consecutive terms notation* as

$$\mathbb{N} = (0, 1, 2, \ldots) = (nl, nl + 1, \ldots, nl + (n - 1))_{l=0}^\infty$$

and expressed as

$$\begin{aligned}\mathbb{N} &= (n\cdot 0=0, n\cdot 0+1=1,\dots,n\cdot 0+(n-1)=n-1,\\ &\quad n\cdot 1=n, n\cdot 1+1=n+1,\dots,n\cdot 1+(n-1)=2n-1,\dots)\end{aligned}$$

Let $\overline{\overline{\mathbb{N}}} = \left(\overline{\overline{m}}_{nk}, \overline{\overline{m}}_{nk+1},\dots,\overline{\overline{m}}_{nk+(n-1)}\right)_{k=0}^{\infty}$ be a rearrangement of the sequence $\mathbb{N}$. Then for any n vectors

$$x^t = \left(x^t_{nl}, x^t_{nl+1},\dots,x^t_{nl+(n-1)}\right)_{l=0}^{\infty} \in l^p\,; \quad t=1,2,\dots,n$$

the n vectors

$$\overline{\overline{x}}^t = \left(x^t_{\overline{\overline{m}}_{nk}}, x^t_{\overline{\overline{m}}_{nk+1}},\cdots,x^t_{\overline{\overline{m}}_{nk+(n-1)}}\right)_{k=0}^{\infty}\,; \quad t=1,2,\dots,n$$

are called *parallel rearrangements* of $x^1, x^2,\cdots,x^n$ respectively.

In [4], we have observed that $\left(l^p, \overline{\overline{\|\cdot,\cdots,\cdot\|}}_p\right)$ is an n-normed space where

$$\overline{\overline{\|x^1,x^2,\cdots,x^n\|}}_p = \sup\{|\overline{\overline{x}}^1,\overline{\overline{x}}^2,\dots,\overline{\overline{x}}^n| : \overline{\overline{x}}^1,\overline{\overline{x}}^2,\dots,\overline{\overline{x}}^n \quad \text{are parallel rearrangements of} \quad x^1,x^2,\cdots,x^n \quad \text{respectively}\} \tag{1}$$

and

$$\left|\overline{\overline{x}}^1,\overline{\overline{x}}^2,\dots,\overline{\overline{x}}^n\right| = \left(\sum_{k=0}^{\infty}\left|det\begin{pmatrix} x^1_{\overline{\overline{m}}_{nk}} & x^1_{\overline{\overline{m}}_{nk+1}} & \cdots & x^1_{\overline{\overline{m}}_{nk+(n-1)}}\\ x^2_{\overline{\overline{m}}_{nk}} & x^2_{\overline{\overline{m}}_{nk+1}} & \cdots & x^2_{\overline{\overline{m}}_{nk+(n-1)}}\\ \cdots & \cdots & \cdots & \cdots\\ x^n_{\overline{\overline{m}}_{nk}} & x^n_{\overline{\overline{m}}_{nk+1}} & \cdots & x^n_{\overline{\overline{m}}_{nk+(n-1)}}\end{pmatrix}\right|^p\right)^{1/p}. \tag{2}$$

Moreover using Minkowski's inequality, we have

$$\left|\overline{\overline{x}}^1,\overline{\overline{x}}^2,\dots,\overline{\overline{x}}^n\right| \le n!\|x^{\pi_1}\|_p\cdot\|x^{\pi_2}\|_\infty\cdots\|x^{\pi_n}\|_\infty;$$

hence

$$\overline{\overline{\|x^1,x^2,\cdots,x^n\|}}_p \le n!\|x^{\pi_1}\|_p\cdot\|x^{\pi_2}\|_\infty\cdots\|x^{\pi_n}\|_\infty; \tag{3}$$

where $\{\pi_1,\pi_2\dots,\pi_n\}$ is any permutation of $\{1,2,\dots,n\}$.

In [1], Malčeski investigated that the function

$$\|x^1,x^2,\cdots,x^n\|_\infty := \sup_{i_1,\dots,i_n}\left|det\begin{pmatrix} x^1_{i_1} & x^1_{i_2} & \dots & x^1_{i_n}\\ x^2_{i_1} & x^2_{i_2} & \dots & x^2_{i_n}\\ \dots & \dots & \dots & \dots\\ x^n_{i_1} & x^n_{i_2} & \dots & x^1_{i_n}\end{pmatrix}\right| \tag{4}$$

defines an n-norm on l^∞ , where $i_1,\dots,i_n\in\mathbb{N}$ and

$$\|x^1,x^2,\cdots,x^n\|_\infty \le n!\cdot\|x^1\|_\infty\cdot\|x^2\|_\infty\cdots\|x^n\|_\infty.$$

But l^p is a subspace of l^∞ therefore we can show that $\|\ldots, .\|_\infty$ forms an n-norm on l^p also.

2. Results

Obviously equivalent n-norms give equivalent derived norms with respect to same linearly independent set. In [4, 5], we have already proved that the two n-norms $\overline{\overline{\|., \ldots, .\|}}_p$ and $\|\ldots, .\|_\infty$ defined on l^p are non-equivalent. While their derived norms with respect to the linearly independent set $\{e^1, \cdots, e^n\}$ are equivalent and equivalent to $\|.\|_\infty$, where $e^t = (\delta_i^t)_{i=0}^\infty$. For details see [5]. Here we shall prove that these two n-norms give equivalent derived norms with respect to many linearly independent sets.

In [7], it has been proved that equivalent n-norms become sequentially equivalent. Here, our aim is to prove that sequentially equivalent n-norms need not be equivalent.

Here we shall use the results of [4, 5] as following lemmas.

Lemma 2.1. For every $x^1, x^2, \cdots, x^n \in l^p$, we have

$$\|x^1, x^2, \cdots, x^n\|_\infty \leq \overline{\overline{\|x^1, x^2, \cdots, x^n\|}}_p. \tag{5}$$

Next, for every $K > 0$ there exists a positive integer N such that $N^{1/p} > K$. Defining $x^t = \left(x_i^t\right)_{i=0}^\infty \in l^p$; $t = 1, 2, \ldots, n$ as follows:

$$x_i^t = \begin{cases} 1 & ; \text{if} \quad i \equiv (t-1)(mod\, n) \quad \text{and} \quad 0 \leq i \leq (Nn-1) \\ 0 & ; \text{otherwise} \end{cases}$$

then we get

$$\overline{\overline{\|x^1, x^2, \cdots, x^n\|}}_p = N^{1/p} > K$$

while

$$\|x^1, x^2, \cdots, x^n\|_\infty = 1.$$

hence we have the following lemma.

Lemma 2.2. The two n-norms $\|\cdot, \cdots, \cdot\|_\infty$ and $\overline{\overline{\|\cdot, \cdots, \cdot\|}}_p$ are non-equivalent.

Theorem 2.3. Let $z^1, z^2, \cdots, z^n \in l^p$ are linearly independent such that the number of non-zero terms of each sequence $z^t \leq \lambda$ for all $t = 1, 2, \cdots, n$ then for all $x \in l^p$, we have

$$\|x\|_\infty^d \leq \overline{\overline{\|x\|}}_p^d \leq \lambda \cdot \|x\|_\infty^d$$

where $\|x\|_\infty^d = max\left\{\|x, z^{t_1}, \ldots, z^{t_{n-1}}\|_\infty : \{t_1, \ldots, t_{n-1}\} \subset \{1, \ldots, n\}\right\}$ and $\overline{\overline{\|x\|}}_p^d = max\left\{\overline{\overline{\|x, z^{t_1}, \ldots, z^{t_{n-1}}\|}}_p : \{t_1, \ldots, t_{n-1}\} \subset \{1, \ldots, n\}\right\}$.

Proof. From lemma 2.1 it is clear that for all $x \in l^p$, we have

$$\|x\|_\infty^d \leq \overline{\overline{\|x\|}}_p^d.$$

Again since the number of non-zero terms of each sequence $z^t \leq \lambda$ therefore for every rearrangement $\overline{\overline{\mathbb{N}}} = \left(\overline{\overline{m}}_{nk}, \overline{\overline{m}}_{nk+1}, \ldots, \overline{\overline{m}}_{nk+(n-1)}\right)_{k=0}^\infty$ of the sequence $\mathbb{N}$ at most λ terms of the series (2) may be non-zero. therefore in view of (2) and (4), we have

$$\overline{\overline{\|x\|}}_p^d \leq \lambda \cdot \|x\|_\infty^d.$$

Hence, we have the theorem.

Corollary 2.4. In general, if $z^1, z^2, \cdots, z^n \in C_{00} \subset l^p$ are n linearly independent vectors then the norms derived by the non-equivalent n-norms $\|\cdot, \cdots, \cdot\|_\infty$ and $\overline{\overline{\|\cdot, \cdots, \cdot\|}}_p$ with respect to linearly independent set $\left\{z^1, z^2, \cdots, z^n\right\}$ are equivalent. Where C_{00} is the space of complex sequences having only finitely many nonzero terms.

In [5], we have already proved that for $z^t = (\delta_i^t)_{i=0}^\infty$ we have

$$\overline{\overline{\|x\|}}_p^d = \|x\|_\infty^d = \|x\|_\infty \qquad \text{for all} \quad x \in l^p. \tag{6}$$

Theorem 2.5. If the sequence $\left(x^l\right)_{l=0}^\infty$ converges to x with respect $\|\cdot, \cdots, \cdot\|_\infty$ then it converges to x with respect to $\overline{\overline{\|\cdot, \cdots, \cdot\|}}_p$ also.

Proof. Let the sequence $\left(x^l\right)_{l=0}^\infty$ converges to x with respect $\|\cdot, \cdots, \cdot\|_\infty$ then it converges to x with respect to its derived norm hence in view of above equation (6) it converges to x with respect $\|\cdot\|_\infty$ also. Hence due to relation (3) it converges to x with respect to $\overline{\overline{\|\cdot, \cdots, \cdot\|}}_p$ also.

In view of lemma 2.1, we have the following theorem.

Theorem 2.6. If the sequence $\left(x^l\right)_{l=0}^\infty$ converges to x with respect $\overline{\overline{\|\cdot, \cdots, \cdot\|}}_p$ then it converges to x with respect to $\|\cdot, \cdots, \cdot\|_\infty$ also.

Thus above results give the following theorem.

Theorem 2.7. Sequentially equivalent n-norms need not be equivalent.

Proof. Combining the theorems 2.5 and 2.6,we see that the two n-norms $\overline{\overline{\|\cdot,\cdots,\cdot\|}}_p$ and $\|\cdot,\cdots,\cdot\|_\infty$ defined on l^p are sequentially equivalent but in view of lemma 2.2, we see that they are non-equivalent.

3. Conclusion

From above discussions, it is clear that equivalence of derived norms need not imply the equivalence of respective n-norms. Moreover, non-equivalent n-norms may derive equivalent norms with respect to many linearly independent set. Further, it has been observed that sequentially equivalent n-norms need not be equivalent n-norms.

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