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Comparison of Summability $|(D,k)(C,\ \alpha,\ \beta)|_p$ and Cesáro $|C,\alpha|_p$ Summability

By

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Abstract

Summability is a branch of mathematical analysis in which an innite series which is usually divergent can converge to a finite sum s (say) by ordinary summation techniques and become summable with the help of deferent summation means or methods. C method was given by Ernesto Cesáro such that ordinary Cesáro summation was written as $(C,1)$ summation whereas generalised Cesáro summation was given as (C,α). In 1913, Hardy [1] proved a theorem on (C,a), $a>0$ summability of the series.

Key Words and Phrases: (D,k) means, (C,α) means, (C,α,b) means, (D,k) (C,α) product means, Fourier Series, Conjugate Series, Lebesgue Integral.
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1. Introduction

Kuttner [2] introduced the summability method (D,α) for functions and investigated some of its properties. Pathak [7] discussed relative strength of summability $|(D,k)(C,l)|_p$ and absolute Cesáro summability. Mishra and Srivastava [6] introduced the Summability method (C,α,β) for functions by generalizing (C,α) summability method. In this paper, we discuss relative strength of summability $|(D,k)(C,\alpha,\beta)|_p$ and absolute Cesáro summability for functions and investigate a relation between different sets of parameters.

2. Some Definitions

Let $f(x)$ be any function which is Lebesgue-measurable, and that $f:[0,+\infty)\to R$, and integrable in $(0,x)$ for any finite x and which is bounded in some right hand neighbourhood of origin. Integrals of the form $\int_0^\infty$ are throughout to be taken

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as $\lim_{x\to\infty}\int_0^x$, $\int_0^x$ being a Lebesgue integral.

Let $k > 0$. If, for $t > 0$, the integral

$$g(t) = g^{(k)}(t) = kt\int_0^\infty \frac{x^{k-1}}{(x+t)^{k+1}} f(x)dx, \tag{2.1}$$

exists and if $g(t) \to s$ as $t \to \infty$, we say that function $f(x)$ is summable (D, k) to the sum s and we write $f(x) \to s(D, k)$ as $x \to \infty$.

We note that, for any fixed $t > 0$, $k > 0$, it is necessary and sufficient for convergence of (2.1) that

$$\int_1^\infty \frac{f(x)}{x^2} dx, \tag{2.2}$$

should converge .

The (C, α, β) transform of $f(x)$, which we denote by $\partial_{\alpha,\beta}(x)$ is given by

$$f(x) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} \frac{1}{x^{\alpha+\beta}} \int_0^x (x-y)^{\alpha-1} y^\beta f(y)dy,\ (\alpha > 0, \beta > -1). \tag{2.3}$$

If this exists for $x > 0$ and $\partial_{\alpha,\beta}(x)$ tends to a limit s as $x \to \infty$, we say that $f(x)$ is summable (C, α, β) to s, and we write $f(x) \to s(C, \alpha, \beta)$. We also write

$$U_{k,\alpha,\beta}(t) = kt\int_0^\infty \frac{x^{k-1}}{(x+t)^{k+1}} \partial_{\alpha,\beta}(x)dx, \tag{2.4}$$

if this exists, and tends to a limit s as $t \to \infty$, we say that the function $f(x)$ is summable $(D, k)(C, \alpha, \beta)$ to s.

When $\beta = 0$, $(D, k)(C, \alpha, \beta)$ and $(D, k)(C, \alpha)$ denote the same method.

If $\alpha \geq 0$, $p \geq 1$, $\beta > -1$, we say that $f(y)$ is summable $|C, \alpha, \beta|_p$ (absolutely summable (C, α, β)) with index p, if

$$\int_T^\infty y^{p-1} \left| \frac{d}{dy} \partial_{\alpha,\beta}(y) \right|^p dy < \infty \quad \text{for some} \quad T \geq 0. \tag{2.5}$$

This is analogue for functions of definition for sequences given by Flett [3]. In any result involving $|C, \alpha, \beta|_p$ for values of $\alpha < 1$, we restrict ourselves to the case in which $f(y)$ is an indefinite Lebesgue integral of a function $a(y)$, say; this ensures that the derivative $\left(\frac{d}{dy}\partial_{\alpha,\beta}(y)\right)$ exists almost everywhere.

Such a restriction is not, however, needed when $\alpha \geq 1$. By analogy with Flett [3], it might at first sight appear and one should define $|C,\alpha,\beta|_p$−summability by

$$\int_0^\infty y^{p-1}\left|\frac{d}{dy}\partial_{\alpha,\beta}(y)\right|^p dy < \infty, (\alpha \geq 0, \beta > -1, p > 1). \tag{2.6}$$

Further suppose that $k > 0$, $\beta > -1$, $\alpha > 0$ and $p \geq 1$. Then we say that the function $f(y)$ is summable $|(D,k)(C,\alpha,\beta)|_p$ or absolutely summable $(D,k)(C,\alpha,\beta)$ with index p, if the integral defined by

$$U_{k,\alpha,\beta}(y) = ky\int_0^\infty \frac{x^{k-1}}{(x+y)^{k+1}}\partial_{\alpha,\beta}(x)dx$$

converges for all $y > 0$, and

$$\int_1^\infty y^{-1}\left|y\frac{d}{dy}U_{k,\alpha,\beta}\right|^p dy < \infty. \tag{2.7}$$

3. Main Results

In this section, we have the following theorems on the relative strength between $|C,\gamma,\beta|_p$ and $|(D,k)(C,\alpha,\beta)|_p$.

Theorem 3.1. Let $\alpha > \gamma \geq 0$, $p \geq 1$, $\beta > -1$. If $f(x)$ is summable $|C,\gamma,\beta|_p$, then it is summable $|C,\alpha,\beta|_p$.

Theorem 3.2. $\alpha \geq 0$, $p \geq 1$, $\gamma \geq 0$. If $f(x)$ is summable $|C,\gamma,\beta|_p$, and the integral defined by $U_{k,\alpha-1,\beta}(y)$ exists for all $y > 0$, then $f(x)$ is summable $|(D,k)(C,\alpha,\beta)|_p$ if $k \leq 1$. Also the convergence of $\int_1^\infty \frac{\partial_{\alpha,\beta}(x)}{x^2}dx$ is implied by $|C,\gamma,\beta|_p$ summability of $f(x)$. We first prove this theorem under unreasonable definition (2.7). However ,if the result holds with (2.7), then it must also hold under the definition of (2.5). This follows from the following two Lemmas.

Lemma 3.1. Let $p \geq 1$, $\gamma > 1$. Suppose that $f(x) \in L(0,x)$ for finite $x > 0$. Suppose that $f(x) \in |C,\gamma,\beta|_p$, according to the definition (2.5). Define

$$\bar{f}(x) = \begin{cases} f(x) & for\ x \geq T \\ 0 & for\ \ x < T \end{cases} \tag{3.1}$$

Let $\bar{\partial}_{\gamma,\beta}(y)$ denote the expression corresponding to $\partial_{\gamma,\beta}(y)$ but with $f(x)$ replaced by $\bar{f}(x)$. Then

$$\int_0^\infty y^{p-1} \left| \frac{d}{dy} \partial_{\gamma,\beta}(y) \right|^p dy < \infty. \tag{3.2}$$

Thus $\bar{f}(x)$ is summable $|C,\gamma,\beta|_p$ under the definition (2.7). (By a result due to Mishra and Mishra [4]).

Lemma 3.2. Let the hypothesis be as in Lemma 3.1,and define $f(x)$ as above. Let $k > 0$, $\beta > -1$ and $\alpha > 0$. Then $|(D,k)(C,\alpha,\beta)|_p$ summability of $\{f(x)\}$ and $\{\bar{f}(x)\}$ are equivalent.

Proof of Lemma 3.1. We are given that, for some $T > 0$,

$$\int_T^\infty x^{p-1} \left| \frac{d}{dx} \partial_{\alpha,\beta}(x) \right|^p dx < \infty. \tag{3.3}$$

But since, if (3.3) holds for given T, it holds for any greater T, it must hold for all sufficiently large T. Now by standard properties of fractional integrals, and since $\gamma > 1$, we have

$$\int_0^T (T-u)^{\gamma-2} u^\beta \left| f(u) \right| \, du < \infty, \tag{3.4}$$

for almost all T (and thus, in particular, for some arbitrary large T), we may thus suppose that T should be chosen so that (3.3) and (3.4) hold. Since $\bar{\partial}_{\gamma,\beta}(x) = 0$ for $x < T$, (3.2) will follow if

$$\int_T^\infty x^{p-1} \left| \frac{d}{dx} \partial_{\gamma,\beta}(x) \right|^p dx < \infty.$$

Since (3.3) holds, this will follow from Minkowskis inequality if we prove that

$$\int_T^\infty x^{p-1} \left| \frac{d}{dx} \left\{ \bar{\partial}_{\gamma,\beta}(x) - \partial_{\gamma,\beta}(x) \right\} \right|^p dx < \infty. \tag{3.5}$$

Now, it follows at once from the definition that, for $x > T$,

$$\bar{\partial}_{\gamma,\beta}(x) - \partial_{\gamma,\beta}(x) = \frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma)\Gamma(\beta+1)} \frac{1}{x^{\gamma+\beta}} \int_0^T (x-y)^{\gamma-1} y^\beta \bar{f}(y) \, dy$$

$$-\frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma)\Gamma(\beta+1)}\frac{1}{x^{\gamma+\beta}}\int_0^T (x-y)^{\gamma-1}y^{\beta}\, f(y)\,dy$$

$$=\frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma)\Gamma(\beta+1)}\frac{1}{x^{\gamma+\beta}}\int_0^T (x-y)^{\gamma-1}\left\{\bar{f}(y)-f(y)\right\}\,dy$$

$$=-\frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma)\Gamma(\beta+1)}\frac{1}{x^{\gamma+\beta}}\int_0^T (x-y)^{\gamma-1}y^{\beta}f(y)\,dy.$$

It follows easily that

$$\frac{d}{dx}\left\{\bar{\partial}_{\gamma,\beta}(x)-\partial_{\gamma,\beta}(x)\right\}=\frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma)\Gamma(\beta+1)}\frac{1}{x^{\gamma+\beta+1}}\times$$

$$\int_0^T [\beta(x-y)+(x-\gamma\,y)]\;(x-y)^{\gamma-2}y^{\beta}f(y)\,dy.$$

For relevant values of variables $|x-\gamma\,y|\ \le x+\gamma\,y\le x+\gamma\,x$, so that

$$\left|\frac{d}{dx}\left\{\bar{\partial}_{\gamma,\beta}(x)-\partial_{\gamma,\beta}(x)\right\}\right|\le\left|\frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma)\Gamma(\beta+1)}\frac{1}{x^{\gamma+\beta+1}}\times\right.$$

$$\left.\int_0^T [\beta(x-y)+(x-\gamma\,y)]\;(x-y)^{\gamma-2}y^{\beta}f(y)\,dy\right|$$

$$\le\frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma)}\frac{(\beta+\gamma+1)\,x}{x^{\gamma+\beta+1}}\int_0^T (x-y)^{\gamma-2}y^{\beta}\,|f(y)|\;dy.$$

If $\gamma\le 2$, then for $x>T$, we have $(x-y)^{\gamma-2}\le(T-y)^{\gamma-2}$, so that

$$\left|\frac{d}{dx}\left\{\bar{\partial}_{\gamma,\beta}(x)-\partial_{\gamma,\beta}(x)\right\}\right|\le\frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma)\Gamma(\beta+1)}\frac{(\beta+\gamma+1)\,x}{x^{\gamma+\beta}}\times$$

$$\int_0^T (x-y)^{\gamma-2}y^{\beta}\,|f(y)|\;dy=\frac{\text{Const.}}{x^{\beta+\gamma}}\qquad\text{by (3.4).}$$

If $\gamma\ge 2$, then $(x-y)^{\gamma-2}\le x^{\gamma-2}$, so that

$$\left|\frac{d}{dx}\left\{\bar{\partial}_{\gamma,\beta}(x)-\partial_{\gamma,\beta}(x)\right\}\right|\le\frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma)\;\Gamma(\beta+1)}\frac{(\beta+\gamma+1)\,x}{x^{\beta+2}}\int_0^T |f(y)|\;dy=\frac{\text{Const.}}{x^{\beta+2}}.$$

Since $\gamma>1$, (3.5) will follow in any case.

Proof of Lemma 3.2. We use notations as in Lemma 3.1, and write further $\bar{U}_{k,\alpha,\beta}(y)$ for the expression corresponding to $U_{k,\alpha,\beta}(y)$ but with $f(x)$ replaced by $\bar{f}(x)$.

We know that for any fixed $y > 0$, $k > 0$, $\beta > -1$, $\alpha > 0$ convergence of $U_{k,\alpha,\beta}(y) = ky \int\limits_0^x \frac{x^{k-1}}{(x+y)^{k+1}} \partial_{\alpha,\beta}(x)\, dx$, is equivalent to the convergence of $\int\limits_1^\infty \frac{\partial_{\alpha,\beta}(x)}{x^2} dx$. Then the conclusion will follow from Minkowskis inequality, if we show that

$$\int\limits_1^\infty y^{p-1} \left| \frac{d}{dy} \left\{ U_{k,\alpha,\beta}(y) - \bar{U}_{k,\alpha,\beta}(y) \right\} \right|^p dy < \infty, \tag{3.6}$$

where we take (3.6) as including the assertion that the integral defined by $U_{k,\alpha,\beta}(y) - \bar{U}_{k,\alpha,\beta}(y)$ converges for all $y > 0$. For large y, we have

$$\partial_{\alpha,\beta}(y) - \bar{\partial}_{\alpha,\beta}(y) = \frac{\Gamma(\gamma+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} \frac{1}{y^{\alpha+\beta}} \int\limits_0^T (y-x)^{\alpha-1} x^\beta f(x)\, dx \tag{3.7}$$

$$= O(1) \frac{1}{y^{\alpha+\beta}} y^{\alpha-1} \int\limits_0^T (y-x)^{\alpha-1} x^\beta \, |f(x)| \, dx$$

$$= O\left(\frac{1}{y^{\alpha+\beta}}\right) \int\limits_0^T x^\beta \, dx$$

$$= O\left(\frac{T}{y^{\alpha+\beta}}\right)^{\beta+1} = O(\tfrac{1}{y})^{\beta+1}, \qquad (T < y).$$

Hence the convergence of

$$ky \int\limits_0^x \frac{x^{k-1}}{(x+y)^{k+1}} \partial_{\alpha,\beta}(x) \left\{ \partial_{\alpha,\beta}(x) - \bar{\partial}_{\alpha,\beta}(x) \right\} dx,$$

follows at once by a result due to (Mishra and mishra [4]) . Now (3.6) is equivalent to

$$\int\limits_1^\infty y^{p-1} dy \left| c \int\limits_0^\infty \frac{x^{k-1}}{(x+y)^{k+2}} (x-ky) \left\{ \partial_{\alpha,\beta}(x) - \bar{\partial}_{\alpha,\beta}(x) \right\} dx \right|^p < \infty. \tag{3.8}$$

Let T_0 be any sufficiently large constant. Then (3.8) will follow from Minkowskis inequality, if we show that

$$\int\limits_1^\infty y^{p-1} dy \left| c \int\limits_0^{T_0} \frac{x^{k-1}}{(x+y)^{k+2}} (x-ky) \left\{ \partial_{\alpha,\beta}(x) - \bar{\partial}_{\alpha,\beta}(x) \right\} dx \right|^p < \infty \tag{3.9}$$

$$\int_1^\infty y^{p-1}dy\left|c\int_{T_0}^\infty \frac{x^{k-1}}{(x+y)^{k+2}}(x-ky)\ \left\{\partial_{\alpha,\beta}(x)-\bar{\partial}_{\alpha,\beta}(x)\right\}\,dx\right|^p<\infty \tag{3.10}$$

For $x<T_0$, we have $|x-k\,y|\ \leq\ x+kx\ \leq\ x(k+1)\ \leq\ T_0(k+1)=C$ (Const.). By (3.9), we have

$$\int_1^\infty y^{p-1}dy\left|c\int_0^{T_0} \frac{x^{k-1}}{(x+y)^{k+2}}(x-ky)\ \left\{\partial_{\alpha,\beta}(x)-\bar{\partial}_{\alpha,\beta}(x)\right\}\,dx\right|^p$$

$$\leq O(1)\int_1^\infty y^{p-1}dy\left|y^{-k-2}\int_0^{T_0} x^{k-1}dx\right|^p$$

$$=O(1)\int_1^\infty y^{p-1}dy\left|y^{-k-2}T_0^k\right|^p$$

$$=O(1)\int_1^\infty y^{-kp-p-1}dy=O(1)\ \left[y^{-kp-p}\right]_1^\infty=O(1).$$

Hence (3.9) follows. By (3.7), the expression on the left of (3.10) does not exceed a constant. Thus by [8]

$$\int_1^\infty y^{p-1}dy\left|c\int_{T_0}^\infty \frac{x^{k-1}}{(x+y)^{k+2}}(x-ky)\ \left\{\partial_{\alpha,\beta}(x)-\bar{\partial}_{\alpha,\beta}(x)\right\}\,dx\right|^p$$

$$=\int_1^\infty y^{p-1}dy\left|c\int_{T_0}^\infty (x+y)^{-2}o(\tfrac{1}{x})^{\beta+1}\,dx\right|^p$$

$$=\int_1^\infty y^{p-1}dy\left|c\int_{T_0}^\infty (x+y)^{-2}o(\tfrac{1}{x})^{\beta+1}\,dx\right|^p$$

$$=O(1)\int_1^\infty y^{p-1}dy\left|\int_{T_0}^\infty (x+y)^{-2}x^{-\beta-1}\,dx\right|^p \tag{3.11}$$

By an obvious change of variables the expression (3.11) is equal to

$$O(1)\int_1^\infty y^{p-1}dy\left|\int_y^\infty t^{-2}(t-y)^{-\beta-1}\,dt\right|^p=O(1)\int_1^\infty y^{\beta\,p-p-1}dy\ =O(1)\,C\ =\ C.$$

The result follows.

Proof of Theorem 3.2. We divide the proof into the following cases.

Case I. $\alpha>\gamma$

Case II. $\alpha = \gamma$

Case III. $\alpha < \gamma$

Here we observe that Case I and II follow from case III, with the aid of Theorem 3.1., for, if $\alpha \geq \gamma$, choose any $\gamma' > \alpha$, summability $|C, \gamma, \beta|_p$ implies summability $|C, \gamma', \beta|_p$ by Theorem 3.1, and it follows from Case III, that this implies $|(D, k)(C, \alpha, \beta)|_p$. Hence it is sufficient to consider the case III only.

Proof of Case III. Since $f(x) \to s(C, \alpha, \beta)$ implies that $f(x) \to s(C, \alpha', \beta)$ for $\alpha' > \alpha > 0$, there is no loss of generality in considering the Case $\gamma = \alpha + k$, k is a positive integer.

We have, by (Mishra & Mishra [4])

$$\frac{d}{dy} U_{k,\alpha,\beta}(y) = C \int_{T_0}^{\infty} \frac{x^{k-1}}{(x+y)^{k+2}} (x - ky)\, \partial_{\alpha,\beta}(x)\, dx. \tag{3.12}$$

Now, by definition

$$\partial_{\alpha+p\,,\beta}(x) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+p+\gamma)(\gamma+\beta+1)} \frac{1}{y^{\alpha+\beta+p}} \int_0^x (x-t)^{\alpha-\gamma+p-1} t^{\gamma+\beta} \partial_{\alpha,\beta}(t) dt.$$

Putting $p = 1$ and $\alpha = \gamma$, we see that

$$\partial_{\alpha+1\,,\beta}(x) = \frac{(\alpha+\beta+1)}{x^{\alpha+\beta+1}} \int_0^x t^{\alpha+\beta} \partial_{\alpha,\beta}(t) dt. \tag{3.13}$$

We also write $R_{\alpha,\beta}(x) = \int_x^{\infty} \frac{\partial_{\alpha,\beta}(t)}{t^2} dt$.

It is clear that, whenever $\int_1^{\infty} \frac{\partial_{\alpha,\beta}(x)}{x^2} dx$ converges, $R_{\alpha,\beta}(x)$ is defined for $x > 0$, and that $R_{\alpha,\beta}(x) \to 0$ as $x \to \infty$. It follows immediately from (3.13) that

$$\partial_{\alpha+1\,,\beta}(x) = -\frac{(\alpha+\beta+1)}{x^{\alpha+\beta+1}} \int_0^x t^{\alpha+\beta} t^2 dR_{\alpha,\beta}(t) dt = O(x^1)$$

and hence that, for $p \geq 1$,

$$\partial_{\alpha+1,\beta}(x) = O(x^1) \tag{3.14}$$

Now by (3.12), we have

$$\frac{d}{dy} U_{k,\alpha,\beta}(y) = C \int_0^{\infty} \frac{x^{k-\alpha-\beta-1}}{(x+y)^{k+2}} (x - ky) x^{\alpha+\beta} \partial_{\alpha,\beta}(x) dx. \tag{3.15}$$

Integrating (3.15) by parts k times, we deduce with the help of (3.14) that

$$\frac{d}{dy}U_{k,\alpha,\beta}(y) = (-1)^k C \int_0^\infty x^{\alpha+\beta+k}\partial_{\alpha+k,\beta}(x)\left\{\frac{d^k}{dx^k}\left[\frac{x^{k-\alpha-\beta-1}}{(x+y)^{k+2}}(x-ky)\right]\right\}dx. \tag{3.16}$$

It is easily verified that the expression in curly brackets (3.16) using result of [10] is

$$O\left(\frac{x^{k-\alpha-\beta-1}}{(x+y)^{k+1}}\right). \tag{3.17}$$

Let

$$R(x,y) = \int_0^x t^{\alpha+\beta+k}\frac{d^k}{dx^k}\left[\frac{t^{k-\alpha-\beta-1}}{(t+y)^{k+2}}(t-ky)\right]dt.$$

In fact, for fixed $k > 0$, we have uniformly in $x > 0$, $y > 0$,

$$R(x,y) = O\left(\frac{x^k}{(x+y)^{k+1}}\right). \tag{3.18}$$

This may be proved by induction on k, if $k = 0$, we have

$$R(x,y) = \int_0^x t^{\alpha+\beta}\left[\frac{t^{k-\alpha-\beta-1}}{(t+y)^{k+2}}(t-ky)\right]dt = \frac{x^k}{(x+y)^{k+1}},$$

hence the result is evident. Suppose that $k \geq 1$, and assume the result true for $k-1$. Integrating by parts, we have

$$R(x,y) = x^{\alpha+\beta+k}\frac{d^{k-1}}{dx^{k-1}}\left[\frac{x^{k-\alpha-\beta-1}}{(x+y)^{k+2}}(x-ky)\right] - (\alpha+\beta+k)\int_0^x t^{\alpha+\beta+k+1}\times$$
$$\frac{\partial^{k-1}}{\partial t^{k-1}}\left\{\frac{t^{k-\alpha-\beta-1}}{(t+y)^{k+2}}(t-ky)\right\}dt.$$

the first term is of required order by (3.17) (with k replaced by $k-1$), and the second by induction hypothesis. Now integrating (3.16) by parts, we have

$$\frac{d}{dy}U_{k,\alpha,\beta}(y) = \int_0^\infty R(x,y)\left(\frac{d}{dx}\partial_{\alpha+k,\beta}(x)\right)dx = \int_0^\infty R(x,y)\left(\frac{d}{dx}\partial_{\gamma,\beta}(x)\right)dx.$$

Since the integrated term tends to 0 as $\partial_{\gamma,\beta}(x)$ is bounded and $R(x,y) \to 0$ as $x \to \infty$. Now we have

$$\left|\frac{d}{dy}U_{k,\alpha,\beta}(y)\right|^p \leq c\left|\int_0^\infty \left\{R(x,y)\,x^{p-1}\right\}^{\frac{1}{p}}\left(\frac{d}{dx}\partial_{\gamma,\beta}(x)\right)\left\{\frac{R(x,y)}{x}\right\}^{\frac{1}{q}}dx\right|^p.$$

Applying Holders inequality with indices p and $\frac{p}{p-1}$, we have

$$\left|\frac{d}{dy}U_{k,\alpha,\beta}(y)\right|^p \leq c\int_0^\infty \left\{R(x,y)\,x^{p-1}\right\}\left|\frac{d}{dx}\partial_{\gamma,\beta}(x)\right|^p \left\{\int_0^\infty \frac{|R(x,y)|}{x}dx\right\}^{p-1}.$$

Using (3.18) and putting $x = t\,y$, we see that the expression in curly brackets

$$\leq C\int_0^x \frac{x^{k-1}}{(x+y)^{k+1}}\,dx = \frac{C}{y}\int_0^x \frac{t^{k-1}}{(1+t)^{k+1}}\,dt = \frac{C}{y},$$

(Since the integral converges). Hence

$$\int_0^\infty y^{p-1}\left|\frac{d}{dy}U_{k,\alpha,\beta}(y)\right|^p \leq \int_0^\infty dy\int_0^\infty x^{p-1}\left|\frac{d}{dx}\partial_{\gamma,\beta}(x)\right|^p |R(x,y)|\,dx$$

$$= C\int_0^\infty x^{p-1}\left|\frac{d}{dx}\partial_{\gamma,\beta}(x)\right|^p dx\,|R(x,y)|\,dy.$$

Again using (3.18), the inner integral

$$\leq C\,x^k\int_0^\infty \frac{1}{(x+y)^{k+1}}\,dy \tag{3.19}$$

on putting $y = x\,t$, the expression on the right of (3.19) is equal to

$$C\int_0^\infty \frac{1}{(1+t)^{k+1}}\,dt = C.$$

Now

$$\begin{aligned}\int_1^\infty \frac{\partial_{\alpha,\beta}(x)}{x^2}dx &= \int_1^x \frac{x^{\alpha+\beta}\partial_{\alpha+\beta}(x)}{x^{\alpha+\beta+2}}dx\\ &= \frac{\partial_{\alpha+1,\beta}(x)}{(\alpha+\beta+1)x} - \frac{\partial_{\alpha+1,\beta}(1)}{(\alpha+\beta+1)} + \frac{(\alpha+\beta+2)}{(\alpha+\beta+1)}\int_1^x \frac{\partial_{\alpha+1,\beta}(x)}{x^2}\,dx.\end{aligned}$$

But we have by [11]

$$\int_1^\infty x^{p-1}\left|\frac{d}{dx}\partial_{\alpha+1,\beta}(x)\right|^p dx < \infty.$$

Also

$$\partial_{\alpha+1,\beta}(x) = \partial_{\alpha+1,\beta}(1) + \int_1^x \left(\frac{d}{dx}\partial_{\alpha+1,\beta}(x)\right)dx. \tag{3.20}$$

By Holders inequality with indices p and q, we have by [9]

$$\left|\int_1^x \left(\frac{d}{dx}\partial_{\alpha+1,\beta}(x)\right)\, dx\right| \leq \left(\int_1^x x^{p-1}\left|\frac{d}{dx}\partial_{\alpha+1,\beta}(x)\right|^p\, dx\right)^{\frac{1}{p}} \left(\int_1^x \frac{1}{x}dx\right)^{\frac{1}{q}} \tag{3.21}$$
$$= O\,(\log x)^{1/q}.$$

From (3.20) and (3.21), we see that $\int\limits_1^\infty \frac{\partial_{\alpha,\beta}(x)}{x^2}\,dx$ is convergent .

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