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Optimal Harvesting in a Tritrophic Food Chain Model with Ratio-Dependent Functional Response

By

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Abstract

The present paper deals with a non-linear mathematical model of three species food chain community having ratio-dependent functional response, where all the species are subjected to optimal harvesting effort with tax as a control instrument to avoid over exploitation of populations. Mathematical analysis of the model equations with regard to the boundedness of solutions, existence of interior equilibrium and their stabilities are carried out. A combined harvesting policy for prey, middle predator and top predator species is discussed by using Pontryagin's Maximum Principle. To verify our mathematical analysis some numerical simulations are carried out.

Keywords : Food chain, Optimal-Harvesting, Ratio-dependent, Boundedness, Taxation.

1. Introduction

In recent years, there is a growing interest in the research field of ratio dependent food chain. In the environment food chains and webs are highly complex and interdependent. Food chain can be modeled by the system of ordinary differential equations that approximate species or functional feeding group behavior with a variety of functional responses. Many simple two species food chain models have been thoroughly explored, while new discoveries continue to be made in examining models with three or four trophic levels (e.g. Moghadas and Gummel 2003). Hsu et al. (2003) discussed a ratio-dependent food chain model and its application in biological control process with Michaelis-Menten type functional response. They proved that food chain model is rich in boundary dynamics and if interior equilibrium point doesnt exist then top predator faces extinction and provide a scenarios for biological control. Further Agarwal and Singh (2011, 2013) analyzed three species ratio dependent food chain model with delay and Michaelis-Menten type and Holling type-II functional response, they found that this model is best use in bio control on pest. The present paper is extension of

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our paper (2011), here we consider the ratio dependent three species food chain model subjected to optimal harvesting effort with tax as a control instrument to avoid over exploitation of populations.

The study of population dynamics with harvesting is a subject of mathematical bio economics and is mainly concerned with the optimal management of renewable resources, (Clark, 1990). Harvesting is commonly practiced in fisheries, forestry and wild life management. Azar, et al., 1995; Kumar, et al., 2002 have investigated the harvesting of predator species predating over two preys. The constant harvesting rate is treated as a control parameter and the system changes its stability to limit cycle when harvesting exceeds a certain limit. The issue that makes these studies difficult is how to drive strategies to maximize the revenues and sustain the populations. Since affecting one population may have unforeseen economic and for ecological consequences for the other, (Armsworth and Roughgarden, 2001). Clark, 1976; assumed two ecologically independent species that experience logistic growth and the harvest rate for each species is proportionally to both its stock level and harvesting effort. This study was extended by MestertonGibbons, 1988; who discussed the optimal approach to equilibrium for such a dynamical system. Srinivasu, et al., 2001, proposed predator-prey Holling type model using harvesting effort as control, and showed that with harvesting it is possible to break the cyclic behavior of system and introduced a globally stable limit cycle in the system. Kar, et al., 2006; has studied the bioeconomic model of a ratiodependent preypredator system with optimal harvesting. They proved that optimal equilibrium populations leads to a situation where total users cost of harvest per unit effort equals the discounted value of the future profit. Later, Leard, et al., 2008; Lenzini, et al., 2009; studied the dynamics of ratio-dependent models that include nonconstant harvesting. Kar, et al. 2010, proposed a biological economic model based on a preypredator dynamics where prey species are continuously harvested and predation is considered type II functional response.

Based on previous results of various authors, we investigate the dynamical behavior of the prey, middle predator and top predator ecosystem due to the variation of economic interest of harvesting, which was discussed in section 2. Conditions for the boundedness, existence of non negative equilibrium, criteria for their local stability are obtained in section 3, 4 and 5. In section 6, a combined harvesting policy for prey, middle predator and top predator species is discussed by using Pontryagin's Maximum Principle. Numerical simulation is done to illustrate the results in section 7. Finally, this paper ends with a conclusion which is presented in section 8.

2. Mathematical Model

In this section, we consider food chain model where predation is governed by ratiodependent functional response. It is assumed that the dynamics of prey population follows logistic growth and is subjected to a dynamic harvesting. To maintain desired level of population, the regulatory agency imposes a tax $s > 0$ (negative value of s denotes subsidy) per unit biomass of the landed prey and predators populations. In the modeling process, $x(t)$, $y(t)$ and $z(t)$ denote the densities of prey, middle predator and super predator population, respectively at any time t in the region under consideration. $E(t)$ represents the combined effort applied to harvest both prey and predators population at time t. Taking note of above, we propose a system dynamics by the set of following non-linear differential equations:

$$\begin{aligned}
\dot{x}(t) &= rx\big(1-\frac{x}{k}\big) - \frac{c_1xy}{a_1y+x} - q_1Ex,\\
\dot{y}(t) &= \frac{m_1xy}{a_1y+x} - d_1y - \frac{c_2yz}{a_2z+y} - q_2Ey,\\
\dot{z}(t) &= -d_2z + \frac{m_2yz}{a_2z+y} - q_3Ez,\\
\dot{E}(t) &= \alpha_0E[(p_1-s)q_1x + (p_2-s)q_2y + (p_3-s)q_3z - c]
\end{aligned} \tag{1}$$

$x(0) > 0$, $y(0) > 0$, $z(0) > 0$ and $E(0) > 0$.

The model parameters are assuming only positive values. It is assumed that prey population follows logistic growth with intrinsic growth rate 'r' and carrying capacity 'k'. For $i = 1, 2, d_i, m_i$ and c_i are natural death rate of predators, fraction of predation term that contributes in predators growth and capturing rates of predators respectively. q_1, q_2 and q_3 are the constant catch ability coefficients for prey, middle predator and top predator population. p_1, p_2 and p_3 are the fixed price per unit of prey and both predator populations respectively, and c is the fixed cost of harvesting population per unit of effort. The constant α_0 is called stiffness parameter measuring the strength of reaction of effort to the perceived rent. The above assumptions are ecologically reasonable and exemplified.

3. Boundedness

In theoretical biology, boundedness of a system implies that the system is biologically well behaved. The following theorem ensures the boundedness of the system (1).

Theorem 3.1. All solutions of system (1) that are initiated in R^4_+ are uniformly bounded into the region $B = \{(x, y, z, E) : 0 \le x \le k, 0 \le W \le \frac{M}{l} + \epsilon$, for any $\epsilon > 0\}$.

4. Equilibrium Analysis

The model system has at most four feasible boundary equilibrium points:

1. The equilibrium points $E_0(0,0,0,0)$ and $E_1(k,0,0,0)$ are obvious.
2. Prey population and middle predator population both can survive in the absence of top predator populations. Hence the equilibrium point $\bar{P}(\bar{x},\bar{y},0,0)$ in $x-y$ plane exists, where $\bar{x}$ and $\bar{y}$ are given by

$$\bar{x} = \frac{m_1 a_1 r - c_1(m_1 - d_1)}{m_1 a_1 r}, \quad \bar{y} = \Big(\frac{m_1 - d_1}{d_1 a_1}\Big)\,\bar{x}. \tag{2}$$

It exists if $m_1 a_1 r > c_1(m_1 - d_1) > 0$.

3. The positive equilibrium point $\tilde{P}(\tilde{x},\tilde{y},\tilde{z},0)$ exists in the first octant, given by

$$\tilde{x} = \frac{k(a_1 A r - c_1(A-1))}{a_1 A r}, \quad \tilde{y} = \frac{(A-1)}{a_1}\,\tilde{x} \quad \tilde{z} = \left(\frac{m_2 - d_2}{a_2 d_2}\right)\tilde{y}, \tag{3a}$$

where $A = \dfrac{m_1}{d_1 + \dfrac{c_2}{m_2 a_2}(m_2 - d_2)}$.

It is found that $\tilde{P}(\tilde{x},\tilde{y},\tilde{z},0)$ exist if

(i) $a_1 A r > c_1(A-1) > 0$

(ii) $m_2 - d_2 > 0$. (3b)

4. Interior equilibrium point $P^*(x^*,y^*,z^*,E^*)$ of system (1) may be obtained by solving the following algebraic equations:

$$r\left(1 - \frac{x}{k}\right) - \frac{c_1 y}{a_1 y + x} - q_1 E = 0, \tag{4}$$

$$\frac{m_1 x}{a_1 y + x} - d_1 - \frac{c_2 z}{a_2 z + y} - q_2 E = 0, \tag{5}$$

$$-d_2 + \frac{m_2 y}{a_2 z + y} - q_3 E = 0, \tag{6}$$

$$\alpha_0(p_1 - s)q_1 x + \alpha_0(p_2 - s)q_2 y + \alpha_0(p_3 - s)q_3 z - \alpha_0 c = 0. \tag{7}$$

Solving equation (5), we get

$$E = \frac{1}{q_2}\left[\frac{m_1 x}{a_1 y + x} - d_1 - \frac{c_2 z}{a_2 z + y}\right] \tag{8}$$

Substituting value of E in equations (4) and (6), we get

$$r\left(1 - \frac{x}{k}\right) - \frac{(c_1 q_1 y + q_1 m_1 x)}{q_2(a_1 y + x)} + \frac{q_1 d_1}{q_2} - \frac{q_1 c_2 z}{q_2(a_2 z + y)} = 0 \tag{9a}$$

$$\frac{m_2 q_2 y + q_3 c_2 z}{q_2(a_2 z + y)} + \frac{q_3 d_1 - d_2 q_2}{q_2} - \frac{q_3 m_1 x}{q_2(a_1 y + x)} = 0. \tag{9b}$$

From equation (7), we have

$$y = \frac{c - (p_1 - s)q_1 x - (p_3 - s)q_3 z}{(p_2 - s)q_2}. \tag{9c}$$

Solving equations (9b) and (9c), we get

$$f(x,z) = 0$$

where

$$\begin{aligned} f(x,z) = & \frac{m_2(c - (p_1 - s)q_1 x - (p_3 - s)q_3 z) + q_3 c_2 z}{a_2(p_2 - s)q_2 z + c - (p_1 - s)q_1 x - (p_3 - s)q_3 z} + \frac{q_3 d_1 - q_2 d_2}{q_2} \\ & - \frac{q_1 m_1 (p_2 - s)q_2 x}{a_1(c - (p_1 - s)q_1 x - (p_3 - s)q_3 z) + (p_2 - s)q_2 x}. \end{aligned} \tag{10a}$$

Also solving equation (9a) and (9c), we get

$$g(x,z) = 0$$

where

$$\begin{aligned} g(x,z) = & \; q_2 r\big(1 - \frac{x}{k}\big) - \frac{c_1 q_1 (c - (p_1 - s)q_1 x - (p_3 - s)q_3 z) + m_1 q_1 (p_2 - s)q_2 x}{a_1(c - (p_1 - s)q_1 x - (p_3 - s)q_3 z) + (p_2 - s)q_2 x} \\ & + q_1 d_1 + \frac{q_1 c_2 (p_2 - s)q_2 z}{a_2(p_2 - s)q_2 z + c - (p_1 - s)q_1 x - (p_3 - s)q_3 z}. \end{aligned} \tag{10b}$$

From (10a), we note the following.

When $z \to 0$, then $x \to x_a$ where

$$x_a = \frac{\{d_1 q_3 + (m_2 - d_2)q_2\}a_1 c}{m_2(p_2 - s)q_2 q_1 + (d_1 q_3 + (m_2 - d_2)q_2)(a_1(p_1 - s)q_1 - (p_2 - s)q_2)} \tag{11a}$$

Clearly $x_a > 0$, if $a_1(p_1 - s)q_1 > (p_2 - s)q_2$ and inequality (3b) hold.

Also from equation (10a), we have

$$\frac{dx}{dz} = \frac{Q_1}{Q_2}.$$

where

$$Q_1 = (q_1(p_1 - s)x - c)((p_2 - s)q_2^2 m_2 a_2 - c_2 q_3)B_1 - (p_3 - s)q_3 m_1 a_1 x B_2$$

$$Q_2 = (p_1 - s)q_1 z\{(p_2 - s)q_2^2 m_2 a_2 - c_2 q_3\}B_1 + \{c - (p_3 - s)q_3 z\}a_1 m_1 B_2.$$

Here

$$B_1 = \frac{1}{[a_2(p_2 - s)q_2 z + c - (p_1 - s)q_1 x - (p_3 - s)q_3 z]^2},$$

$$B_2 = \frac{(p_2 - s)q_2 q_1}{[a_1(c - (p_1 - s)q_1 x - (p_3 - s)q_3 z) + (p_2 - s)q_2 x]^2}.$$

It is clear that $\frac{dx}{dz} > 0$, if either

(i) $Q_1 > 0$ and $Q_2 > 0$,
(ii) $Q_1 < 0$ and $Q_2 < 0$ hold. (11b)

Again from equation (10b), we note that when $z \to 0$, then $x \to x_b$, where

$$x_b = \frac{-A_2 \pm \sqrt{(A_2^2 - 4A_1A_3)}}{2A_1}, \tag{11c}$$

where

$$A_1 = r(p_2 - s)q_2^2 - a_1 r(p_1 - s)q_1q_2 (< 0),$$
$$A_2 = \{(p_1 - s)q_1a_1 - (p_2 - s)q_2\}k(q_2r + q_1d_1) - q_2a_1rc - kq_1q_2m_1(p_2 - s) + kc_1(p_1 - s)q_1^2,$$
$$A_3 = ck\{a_1(q_2r + q_1d_1) - c_1q_1\}(> 0).$$

Clearly $A_3 > 0$ and $A_1 < 0$ if $a_1(p_1-s)q_1 > (p_2-s)q_2$ and $cka_1(q_2r + q_1d_1) - c_1q_1 > 0$ is satisfied.

We also have $\frac{dx}{dz} = -\frac{\frac{\partial g}{\partial z}}{\frac{\partial g}{\partial x}}$, where

$$\frac{\partial g}{\partial x} = (m_1a_1 - c_1)\{q_1(p_2 - s)q_2(c + (p_3 - s)q_3z\}B_1 + c_2(p_1 - s)q_1zB_2,$$
$$\frac{\partial g}{\partial z} = (p_2 - s)q_1(p_3 - s)q_3q_2x(m_1a_1 - c_1)B_1 + .c_2\{c - (p_1 - s)q_1x\}B_2 - \frac{r}{k}.$$

We note that $\frac{dx}{dz} < 0$, if either

(i) $\frac{\partial g}{\partial x} > 0$ and $\frac{\partial g}{\partial z} > 0$,
(ii) $\frac{\partial g}{\partial x} < 0$ and $\frac{\partial g}{\partial z} < 0$ hold. (11d)

From the above analysis we found that two isoclines (10a) and (10b) intersect at a unique (x^*, z^*), if in addition to conditions (2), (3),(11b) and (11d), the inequality $x_a < x_b$ holds. Knowing the value of x^* and z^*, the value of y^* and E^* can be obtained as

$$y^* = \frac{c - (p_1 - s)q_1x^* - (p_3 - s)q_3z^*}{(p_2 - s)q_2},$$
$$E^* = \frac{1}{q_2}\left[\frac{m_1x^*}{a_1y^* + x^*} - d_1 - \frac{c_2z^*}{a_2z^* + y^*}\right]. \tag{11e}$$

For x^*, y^*, z^* and E^* to be positive, we must have

$$a_1(p_1 - s)q_1 > (p_2 - s)q_2, \quad m_2 > d_2 \text{ and } c > (p_2 - s)q_2y^* + (p_3 - s)q_3z^*.$$

5. Stability of Equilibrium States

5.1. Local Stability Analysis

The local stability of the equilibrium states of nonlinear model system (1) can be analyzed by the corresponding linearized system which obtained by linearizing the system (1) in the vicinity of equilibrium points. The nature of the eigenvalues of the Jacobian matrix corresponding to each equilibrium points determined the local stability of the each equilibrium. The Jacobian matrix of the system (1) is:

$$V(x,y,z,E) = \begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix},$$

where

$$a_{11} = r\left(1 - \frac{x}{k}\right) - \frac{c_1 y}{a_1 y + x} - q_1 E + x\left(\frac{-r}{k} + \frac{c_1 y}{(a_1 y + x)^2}\right),$$

$$a_{12} = \frac{-c_1 x^2}{(a_1 y + x)^2}, \qquad a_{14} = -q_1 x, \qquad a_{21} = \frac{m_1 a_1 y^2}{(a_1 y + x)^2},$$

$$a_{22} = \frac{m_1 x}{a_1 y + x} - d_1 - \frac{c_2 z}{a_2 z + y} - q_2 E + \left[\frac{-m_1 a_1 x y}{(a_1 y + x)^2} + \frac{c_2 y z}{(a_2 z + y)^2}\right],$$

$$a_{23} = -\frac{c_2 y^2}{(a_2 z + y)^2}, \qquad a_{24} = -q_2 y, \qquad a_{32} = \frac{m_2 a_2 z^2}{(a_2 z + y)^2},$$

$$a_{33} = \left(-d_2 + \frac{m_2 y}{a_2 z + y} - q_3 E\right) - \frac{m_2 a_2 y z}{(a_2 z + y)^2},$$

$$a_{34} = -q_3 z, \qquad a_{41} = \alpha_0 (p_1 - s) q_1 E, \qquad a_{42} = \alpha_0 (p_2 - s) q_2 E,$$

$$a_{43} = \alpha_0 (p_3 - s) q_3 E,$$

$$a_{44} = \alpha_0 (p_1 - s) q_1 x + \alpha_0 (p_2 - s) q_2 y + \alpha_0 (p_3 - s) q_3 z - \alpha_0 c.$$

Now we consider the various equilibrium states separately:

For $\bar{P}(\bar{x}, \bar{y}, 0, 0)$, we note that when the inequality$(p_1 - s)q_1\bar{x} + (p_2 - s)q_2\bar{y} > c$ and $m_2 - d_2 > 0$ holds, then $\bar{P}$ becomes unstable in $z - E$ plane. The other two eigenvalue are the roots of equation

$$\lambda^2 + \left(\frac{r\bar{x}}{k} + \frac{(m_1 a_1 - c_1)\bar{x}\bar{y}}{(a_1\bar{y} + \bar{x})^2}\right)\lambda + \frac{m_1 a_1 r^2 \bar{x}\bar{y}}{(a_1\bar{y} + \bar{x})^2 k} = 0.$$

Since $m_1 > c_1$, thus roots of eigenvalues corresponding to x and $y-$directions have negative real parts. Hence $\bar{P}$ has a stable manifold in $x - y$ plane and unstable manifold in $z - E$ plane.

Further, the eigen value for $\tilde{P}(\tilde{x}, \tilde{y}, \tilde{z}, 0)$ are the zeros of the polynomial

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$$

where

$$b_1 = \frac{r\tilde{x}}{k} + \frac{(m_1a_1 - c_1)\tilde{x}\tilde{y}}{(a_1\tilde{y} + \tilde{x})^2} + \frac{(m_2a_2 - c_2)\tilde{y}\tilde{z}}{(a_2\tilde{z} + \tilde{y})^2} \ (> 0),$$

$$b_2 = \frac{a_1m_1a_2m_2\tilde{x}\tilde{y}^2\tilde{z} - c_1\tilde{y}^2\tilde{z}(a_2m_2 - c_2)}{(a_1\tilde{y} + \tilde{x})^2(a_2\tilde{z} + \tilde{y})^2} + \frac{r\tilde{x}\tilde{y}\tilde{z}(a_2m_2 - c_2)}{k(a_2\tilde{z} + \tilde{y})^2}$$
$$+ \frac{a_1m_1r\tilde{x}^2\tilde{y}}{k(a_1\tilde{y} + \tilde{x})^2} + \frac{a_1m_1c_1\tilde{x}\tilde{y}^2(\tilde{x} - 1)}{(a_1\tilde{y} + \tilde{x})^4} \ (> 0),$$

$$b_3 = \frac{a_1m_1a_2m_2r\tilde{x}^2\tilde{y}^2\tilde{z}}{k(a_1\tilde{y} + \tilde{x})^2(a_2\tilde{z} + \tilde{y})^2} + \frac{a_1m_1a_2m_2c_1\tilde{x}\tilde{y}^3\tilde{z}(\tilde{x} - 1)}{(a_1\tilde{y} + \tilde{x})^4(a_2\tilde{z} + \tilde{y})^2} \ (> 0).$$

Thus, from the Routh-Hurwitz criteria, the necessary and sufficient condition for $\tilde{P}$ to be asymptotically stable in $x-y-z$ plane if $b_1 > 0$, $b_3 > 0$ and $b_1b_2 - b_3 > 0$. But one of the eigen value in $E-$direction is $\lambda = \alpha_0(p_1 - s)q_1\tilde{x} - \alpha_0c + \alpha_0(p_2 - s)q_2\tilde{y} + \alpha_0(p_3 - s)q_3\tilde{z}$, which is positive, so equilibria $\tilde{P}$ is always unstable in $E-$direction.

Finally we investigate the local stability of interior equilibrium $P^*(x^*, y^*, z^*, E^*)$. We first find the variational matrix $V(P^*)$ at interior equilibrium point P^* is

$$V(P^*) = \begin{bmatrix} m_{11} & m_{12} & 0 & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & 0 \end{bmatrix},$$

where

$$m_{11} = x^*\left(\frac{-r}{k} + \frac{c_1y^*}{(a_1y^* + x^*)^2}\right), \quad m_{12} = \frac{-c_1x^{*2}}{(a_1y^* + x^*)^2}, \quad m_{14} = -q_1x^*,$$

$$m_{21} = \frac{m_1a_1y^{*2}}{(a_1y^* + x^*)^2}, \ m_{22} = \frac{-m_1a_1x^*y^*}{(a_1y^* + x^*)^2} + \frac{c_2y^*z^*}{(a_2z^* + y^*)^2}, \ m_{23} = -\frac{c_2y^{*2}}{(a_2z^* + y^*)^2},$$

$$m_{24} = -q_2y^*, \quad m_{32} = \frac{m_2a_2z^{*2}}{(a_2z^* + y^*)^2}, \ m_{33} = -\frac{m_2a_2y^*z^*}{(a_2z^* + y^*)^2}, \ m_{34} = -q_3z^*,$$

$$m_{41} = \alpha_0(p_1 - s)q_1E^*, \quad m_{42} = \alpha_0(p_2 - s)q_2E^*, \quad m_{43} = \alpha_0(p_3 - s)q_3E^*.$$

The eigenvalues of above matrix are the roots of the characteristic polynomial

$$\lambda^4 + M_1\lambda^3 + M_2\lambda^2 + M_3\lambda + M_4 = 0 \qquad (12)$$

where

$$
\begin{aligned}
M_1 =\ & -(m_{11}+m_{22}+m_{33}),\\
M_1 =\ & \frac{rx^*}{k}+\frac{(a_1m_1-c_1)x^*y^*}{(a_1y^*+x^*)^2}+\frac{(a_2m_2-c_2)y^*z^*}{(a_2z^*+y^*)^2}\ (>0),\\
M_2 =\ & m_{11}m_{22}+m_{11}m_{33}+m_{22}m_{33}-m_{12}m_{21}-m_{23}m_{32}-m_{34}m_{43}-m_{14}m_{41}\\
& -m_{24}m_{42},\\
M_2 =\ & \frac{rx^*(m_2a_2-c_2)y^*z^*}{k(a_2z^*+y^*)^2}+\frac{ra_1m_1x^{*2}y^*}{k(a_1y^*+x^*)^2}+\frac{\{m_2a_2(m_1a_1-c_1)+c_1c_2\}x^*y^{*2}z^*}{(a_1y^*+x^*)^2(a_2z^*+y^*)^2}\\
& +\alpha_0(p_1-s)q_1^2x^{*2}+\alpha_0(p_2-s)q_2^2y^{*2}+\alpha_0(p_3-s)q_3^2z^{*2}\ (>0),\\
M_3 =\ & -m_{12}m_{24}m_{41}+m_{11}m_{23}m_{32}+m_{12}m_{21}m_{33}-m_{11}m_{22}m_{33}+m_{11}m_{34}m_{43}\\
& +m_{22}m_{34}m_{43}+m_{14}m_{22}m_{41}+m_{14}m_{33}m_{41}-m_{14}m_{21}m_{42}+m_{11}m_{24}m_{42}\\
& +m_{24}m_{33}m_{42}-m_{23}m_{34}m_{42}-m_{24}m_{32}m_{43}\\
M_3 =\ & \frac{a_2m_2ra_1m_1x^{*2}y^{*2}z^*}{k(a_1y^*+x^*)^2(a_2z^*+y^*)^2}+\frac{q_3^2(p_3-s)x^*y^*z^{*2}\alpha_0(a_1m_1-c_1)}{(a_1y^*+x^*)^2}\\
& +\frac{q_1^2(p_1-s)x^{*2}y^*z^*\alpha_0(a_2m_2-c_2)}{(a_2z^*+y^*)^2}+\frac{\{q_3^2(p_3-s)z^{*2}+q_2^2(p_2-s)y^{*2}\}r\alpha_0x^*}{k}\\
& +\frac{\{q_1(p_1-s)x^{*2}+q_2(p_2-s)y^{*2}\}x^*y^*\alpha_0(a_1m_1q_1-c_1q_2)}{(a_1y^*+x^*)^2}\\
& +\frac{\{q_2(p_2-s)y^{*2}+q_3(p_3-s)z^{*2}\}y^*z^*\alpha_0(a_2m_2q_2-c_2q_3)}{(a_2z^*+y^*)^2}\ (>0)\\
M_4 =\ & m_{12}m_{21}m_{34}m_{43}-m_{11}m_{22}m_{34}m_{43}+m_{12}m_{24}m_{33}m_{41}-m_{11}m_{24}m_{33}m_{42}\\
& +m_{11}m_{23}m_{34}m_{42}-m_{12}m_{23}m_{34}m_{41}+m_{14}m_{23}m_{32}m_{41}-m_{14}m_{22}m_{33}m_{41}\\
& +m_{14}m_{21}m_{33}m_{42}-m_{14}m_{21}m_{32}m_{43}+m_{11}m_{24}m_{32}m_{43},\\
M_4 =\ & \frac{q_3^2(p_3-s)r\alpha_0a_1m_1x^{*2}y^*z^{*2}}{k(a_1y^*+x^*)^2}+\frac{q_3(p_3-s)r\alpha_0x^*y^*z^{*3}(a_2m_2q_2-c_2q_3)}{k(a_2z^*+y^*)^2}
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{\{q_1(p_1-s)x^{*2}+q_3(p_3-s)z^{*2}+q_2(p_2-s)y^{*2}\}\alpha_0a_2m_2x^*y^{*2}z^*(a_1m_1q_1-c_1q_2)}{(a_1y^*+x^*)^2(a_2z^*+y^*)^2}\\
& +\frac{q_3\alpha_0c_1c_2x^*y^{*2}z^*\{q_1(p_1-s)x^{*2}y^*+q_3(p_3-s)z^{*2}\}}{(a_1y^*+x^*)^2(a_2z^*+y^*)^2}+\frac{q_2^2(p_2-s)r\alpha_0a_2m_2x^*y^{*3}z^*}{k(a_2z^*+y^*)^2}
\end{aligned}
$$

(>0).

Now in view of above calculation, we have following result.

Theorem 5.1. Equilibrium point $P^*(x^*,y^*,z^*,E^*)$ is locally asymptotically stable if and only if

(i) $a_1m_1q_1 > c_1q_2$, $a_2m_2q_2 > c_2q_3$, $a_1m_1-c_1>0$ and $a_2m_2-c_2>0$.

(ii) $M_4 > 0$, $M_2 > 0$, $M_1 > 0$.
(iii) $M_3(M_1M_2 - M_3) > M_1^2 M_4$.

This theorem directly follows from the Routh-Hurwitz criterion. Hence P^* is locally asymptotically stable.

6. Optimal Harvesting Policy

In this section, the optimal harvesting policy is discussed which plans to maximize the total discounted net revenue from the harvesting using taxation as control instrument.

The net economic revenue to society $\pi(x, y, z, E, s, t)$ = net revenue to regulatory agency + net revenue to the harvester

$$\pi(x, y, z, E, s, t) = (p_1q_1x + p_2q_2y + p_3q_3z - c)E.$$

Our objective is to solve the problem

$$\max \int_0^\infty \pi(x, y, z, E, s, t)e^{-\delta t}\, dt$$

subjected to state equations of (1) and to the control constraint

$$s_{\min} \le s \le s_{\max} \tag{13a}$$

where δ is the instantaneous annual rate of discount. To solve above problem, we use Pontryagins Maximum Principle. The associated Hamiltonian function is given by

$$\begin{aligned}
&H(x, y, z, E, s, t) = \\
&e^{-\delta t}(p_1q_1x + p_2q_2y + p_3q_3z - c)E + \lambda_1(t)\left[rx\left(1 - \frac{x}{k}\right) - \frac{c_1xy}{a_1y + x} - q_1Ex\right] \\
&+\lambda_2(t)\left[\frac{m_1xy}{a_1y + x} - d_1y - \frac{c_2yz}{a_2z + y} - q_2Ey\right] + \lambda_3(t)\left[-d_2z + \frac{m_2yz}{a_2z + y} - q_3Ez\right] \\
&+\lambda_4(t)\alpha_0E\{(p_1 - s)q_1x + (p_2 - s)q_2y + (p_3 - s)q_3z - c\},
\end{aligned}$$

where λ_1, λ_2, λ_3 and λ_4 are adjoint variables.

For H to be maximum on the control set, we must have $\frac{\partial H}{\partial s} = 0$, which implies that

$$\lambda_4(t) = 0. \tag{13b}$$

Now from maximum principle, we have

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial y}, \quad \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial z}, \quad \frac{d\lambda_4}{dt} = -\frac{\partial H}{\partial E}.$$

From above equations and using (13b), we get

$$e^{-\delta t}(p_1q_1x + p_2q_2y + p_3q_3z - c) = \lambda_1q_1x + \lambda_2q_2y + \lambda_3q_3z. \tag{14a}$$

Now, considering interior equilibrium point P^*, we have

$$\frac{d\lambda_1}{dt} = -e^{-\delta t}p_1q_1E^* + \lambda_1(t)x^*\left[\frac{r}{k} - \frac{c_1y^*}{(a_1y^*+x^*)^2}\right] + \lambda_2(t)\frac{a_1m_1{y^*}^2}{(a_1y^*+x^*)^2} \tag{14b}$$

$$\frac{d\lambda_2}{dt} = -e^{-\delta t}p_2q_2E^* + \lambda_1(t)\left[\frac{c_1x^{*2}}{(a_1y^*+x^*)^2}\right] + \lambda_2(t)\left[\frac{a_1m_1x^*y^*}{(a_1y^*+x^*)^2} - \frac{c_2y^*z^*}{(a_2z^*+y^*)^2}\right]$$
$$+\lambda_3(t)\left[\frac{m_2a_2z^{*2}}{(a_2z^*+y^*)^2}\right] \tag{14c}$$

$$\frac{d\lambda_3}{dt} = -e^{-\delta t}p_3q_3E^* + \lambda_2(t)\left[\frac{c_2y^{*2}}{(a_2z^*+y^*)^2}\right] + \lambda_3(t)\left[\frac{m_2a_2y^*z^*}{(a_2z^*+y^*)^2}\right]. \tag{14d}$$

On solving equations (14b) and (14c) by using (14a), we get

$$\frac{d^2\lambda_1}{dt^2} - \left(\frac{rx^*}{k} - \frac{c_1x^*y^*}{(a_1y^*+x^*)^2} + \frac{m_1a_1x^*y^*}{(a_1y^*+x^*)^2} + \frac{(m_2a_2q_2 - c_2q_3)y^*z^*}{q_3(a_2z^*+y^*)^2}\right)\frac{d\lambda_1}{dt}$$
$$-\left[\frac{m_1a_1c_1x^{*2}y^{*2}}{(a_1y^*+x^*)^4} - \frac{m_1a_1q_1m_2a_2x^*y^{*2}z^*}{q_3(a_2z^*+y^*)^2(a_1y^*+x^*)^2} - \left(\frac{rx^*}{k} - \frac{c_1x^*y^*}{(a_1y^*+x^*)^2}\right).\right.$$
$$\left.\left(\frac{m_1a_1x^*y^*}{(a_1y^*+x^*)^2} + \frac{(m_2a_2q_2 - c_2q_3)y^*z^*}{q_3(a_2z^*+y^*)^2}\right)\right]\lambda_1 = e^{-\delta t}S_1 \tag{14e}$$

where

$$S_1 = p_1q_1E^*\delta + p_1q_1E^*\left(\frac{m_1a_1x^*y^*}{(a_1y^*+x^*)^2} + \frac{(m_2a_2q_2 - c_2q_3)y^*z^*}{q_3(a_2z^*+y^*)^2}\right)$$
$$+\frac{m_1a_1y^{*2}}{(a_1y^*+x^*)^2}\left[p_2q_2E^* + \frac{m_2a_2z^*(p_1q_1x^* + p_2q_2y^* + p_3q_3z^* - c)}{q_3(a_2z^*+y^*)^2}\right].$$

The complete solution of (14e) is of the form

$$\lambda_1(t) = A_1e^{-v_1t} + A_2e^{-v_2t} + e^{-\delta t}\frac{S_1}{D}, \tag{14f}$$

where $A_i (i = 1, 2)$ are arbitrary constants and $v_i (i = 1, 2)$ are the roots of auxillary equation and

$$D = \delta^2 + \left(\frac{rx^*}{k} + \frac{(m_1a_1 - c_1)x^*y^*}{(a_1y^*+x^*)^2} + \frac{(m_2a_2q_2 - c_2q_3)y^*z^*}{q_3(a_2z^*+y^*)^2}\right)\delta + \left[\frac{m_1a_1c_1x^{*2}y^{*2}}{(a_1y^*+x^*)^4}\right.$$
$$-\frac{m_1a_1q_1m_2a_2x^*y^{*2}z^*}{q_3(a_2z^*+y^*)^2(a_1y^*+x^*)^2} + \left(\frac{rx^*}{k} - \frac{c_1x^*y^*}{(a_1y^*+x^*)^2}\right)\left(\frac{m_1a_1x^*y^*}{(a_1y^*+x^*)^2}\right.$$
$$\left.\left.+\frac{(m_2a_2q_2 - c_2q_3)y^*z^*}{q_3(a_2z^*+y^*)^2}\right)\right] \neq 0$$

It is clear from (14d) that λ_1 is bounded iff $v_i < 0 (i = 1, 2)$ or $A_i\,(i = 1, 2)$ are identically zero. For the robust calculations we take $A_i = 0$, $(i = 1, 2)$ and ignore that $v_i < 0\,(i = 1, 2)$. Thus we have

$$\lambda_1(t) = e^{-\delta t}\frac{S_1}{D} \tag{15a}$$

$$\lambda_2(t) = e^{-\delta t}\frac{S_2}{D}, \tag{15b}$$

where

$$S_2 = \delta\left(p_1q_1E^* + \frac{m_2a_2z^*(p_1q_1x^* + p_2q_2y^* + p_3q_3z^* - c)}{q_3(a_2z^* + y^*)^2}\right) - \frac{p_1q_1c_1E^*x^{*2}}{(a_1y^* + x^*)^2}$$
$$+\frac{m_2a_2p_1q_1^2E^*x^*z^*}{q_3(a_2z^* + y^*)^2} + \left(\frac{rx^*}{k} - \frac{c_1x^*y^*}{(a_1y^* + x^*)^2}\right)\Bigg(p_1q_1E^*$$
$$+\frac{m_2a_2z^*(p_1q_1x^* + p_2q_2y^* + p_3q_3z^* - c)}{q_3(a_2z^* + y^*)^2}\Bigg).$$

Again solving equation (14b) and (14d) by using (14a), and proceeding in a similar manner, we have

$$\lambda_3(t) = e^{-\delta t}\frac{S_3}{D_1} \tag{15c}$$

where

$$S_3 = \delta\left(p_1q_1E^* - \frac{c_2y^*(p_1q_1x^* + p_2q_2y^* + p_3q_3z^* - c)}{q_2(a_2z^* + y^*)^2}\right) - \Bigg(\frac{rx^*}{k}$$
$$-\frac{(c_1q_2 + a_1m_1q_1)x^*y^*}{q_2(a_1y^* + x^*)^2}\Bigg)\left(-p_1q_1E^* + \frac{c_2y^*(p_1q_1x^* + p_2q_2y^* + p_3q_3z^* - c)}{q_2(a_2z^* + y^*)^2}\right)$$
$$-\frac{q_1c_2x^*y^*}{q_2(a_2z^* + y^*)^2}\left(-p_1q_1E^* - \frac{a_1m_1y^*(p_1q_1x^* + p_2q_2y^* + p_3q_3z^* - c)}{q_2(a_1y^* + x^*)^2}\right),$$

$$D_1 = \delta^2 + \left(\frac{rx^*}{k} + \frac{(a_2m_2q_2 - c_2q_3)z^*y^*}{q_2(a_2z^* + y^*)^2} - \frac{(a_1m_1q_1 + c_1q_2)x^*y^*}{q_2(a_1y^* + x^*)^2}\right)\delta$$
$$+\frac{a_1m_1q_1q_3c_2x^*z^*y^{*2}}{q_2^2(a_2z^* + y^*)^2(a_1y^* + x^*)^2} + \left(\frac{rx^*}{k} - \frac{(a_1m_1q_1 + c_1q_2)x^*y^*}{q_2(a_1y^* + x^*)^2}\right)$$
$$\frac{(a_2m_2q_2 - c_2q_3)z^*y^*}{q_2(a_2z^* + y^*)^2}.$$

Substituting the value of λ_1, λ_2 and λ_3 into (14a), we get

$$\left(p_1 - \frac{S_1}{D}\right)q_1x^* + \left(p_2 - \frac{S_2}{D}\right)q_2y^* + \left(p_3 - \frac{S_3}{D_1}\right)q_3z^* = c. \tag{16a}$$

For an optimal effort, we have

$$E^* = \frac{1}{q_1}\left[r\left(1 - \frac{x^*}{k}\right) - \frac{c_1y^*}{a_1y^* + x^*}\right] = \frac{1}{q_2}\left[\frac{m_1x^*}{a_1y^* + x^*} - d_1 - \frac{c_2z^*}{a_2z^* + y^*}\right]$$

$$= \frac{1}{q_3}\left[- d_2 + \frac{m_2 y^*}{a_2 z^* + y^*}\right]. \tag{16b}$$

Equations (16a) and (16b) gives the optimal equilibrium levels of populations

$$x^* = x_\delta, \qquad y^* = y_\delta \quad \text{and} \quad z^* = z_\delta$$

When $\delta \to +\infty$, it can easily seen that

$$\left(\frac{S_1}{D}\right), \left(\frac{S_2}{D}\right), \left(\frac{S_3}{D_1}\right) \to 0$$

which imply

$$p_1 q_1 x_\infty + p_2 q_2 y_\infty + p_3 q_3 z_\infty = c. \tag{17}$$

Then the optimal equilibrium levels of effort and tax are given by

$$E_\delta = \frac{1}{q_2}\left[\frac{m_1 x_\delta}{a_1 y_\delta + x_\delta} - d_1 - \frac{c_2 z_\delta}{a_2 z_\delta + y_\delta}\right],$$

$$s_\delta = \frac{p_1 q_1 x_\delta + p_2 q_2 y_\delta + p_3 q_3 z_\delta - c}{q_1 x_\delta + q_2 y_\delta + q_3 z_\delta}.$$

From the above analysis, we can observe following points:

1. $\lambda_i(t)e^{\delta t}$ $(i = 1, 2, 3, 4)$ remain bounded as $t \to \infty$, hence they satisfy the transversality condition at ∞.
2. From(14a), we have

$$\lambda_1 q_1 x + \lambda_2 q_2 y + \lambda_3 q_3 z = e^{-\delta t} \frac{\partial \pi}{\partial E}\bigg|_{\text{at } P^*}$$

   Thus, at the steady state the total users cost of harvest per unit effort is equal to the discounted value of the future price.
3. From equation (17) it can seen that $p_1 q_1 x_\infty + p_2 q_2 y_\infty + p_3 q_3 z_\infty - c \to 0$ when $\delta \to +\infty$.

Hence, the net economic revenue is zero when discounting factor is infinitely large.

7. Numerical Simulation

To facilitate the interpretation of our mathematical findings by numerical simulation, we integrate system (1) using fourth order Runga-kutta method under the following set of compatible parameters with the help of MATLAB Software package. Consider the following set of parameter values to study system (1), numerically.

$\alpha_0 = 1$, $k = 1.1$, $m_1 = 3.5$, $m_2 = 2.5$, $c_1 = 3.5$, $c_2 = 0.3$, $p_1 = 8$, $p_2 = 7$, $p_3 = 6$, $a_2 = 1$, $q_1 = 4$, $q_2 = 0.52$, $q_3 = 1$, $d_1 = 2$, $d_2 = 1.5$, $c = 11$, $\delta = 1.5$, $r = 3.3$, $s = 2$.

For the above set of parameter values, we find that all the equilibrium point for the system exists and given by $\bar{P}(0.61, 0.450, 0, 0)$, $\tilde{P}(0.640, 0.41661, 0.27774, 0)$, $P^*(0.412247, 0.24482, 0.117384, 0.18794)$.

It is found that all the conditions of theorem (5.1) for local stability are satisfied and roots of characteristic equation becomes $-0.26252 \pm 86432i$, $-0.596829 \pm 0.261933i$, which shows that, unique positive equilibrium $P^*(x^*, y^*, z^*, E^*)$ is locally asymptotically stable.

Now for different values of tax(s), we have following results,

s	x^*	y^*	z^*	E^*
0	0.3101285	0.1766296	0.0721595	0.2748939
1	0.3539211	0.2052380	0.0899172	0.2383907
2	0.4122472	0.2448199	0.1173836	0.1879399
4	0.6166476	0.3976784	0.2566302	0.0194603
6	1.3035510	1.0992234	1.8025071	-0.552959

From above table, we see that the harvesting effort E^* decreases while the corresponding level of prey x, middle predator y and top predator z population increases as the tax s increases. There exists a value of tax $(4 < s < 6)$ imposed by regulatory agency, for which the equilibrium effort level becomes zero and in this case prey-middle-top predator populations remain unexploited. Also the optimal equilibrium level of prey-middle-top predator population, harvesting effort and tax are obtained as

$$x_\delta = 0.45669, \ y_\delta = 0.2714601, \ z_\delta = 0.130598, \ E_\delta = 0.18790, \ s_\delta = 2.5665.$$

Figures have been plotted between dependent variables and time for different parameter values to shows changes occurring in population with time under different conditions. The results of numerical simulation are displayed graphically. In figure (1) the prey, first predator top predator populations and harvesting effort are plotted against time. From figure it is noted for given initial values both the populations tend to their corresponding value of equilibrium point P^* and hence coexist in the form of steady state assuring local stability of P^*.

From figure 2(a-d), we can depict that prey, middle predator population and top predator population increases with the increase in s, and finally attain their equilibrium levels. This is obvious as the tax increases harvesting effort for population decreases and then the population of prey, middle predator and hence top predator population increases. For $s = 4.1455$ harvesting effort E become zero.

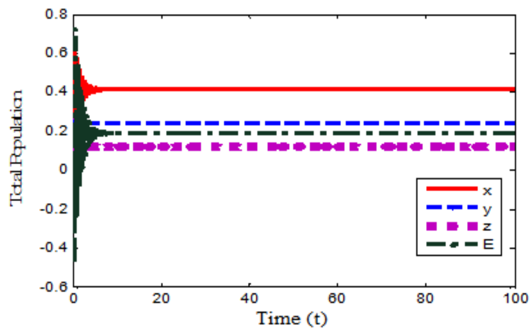

Fig. 1. Stable behaviour of u, x, y and z

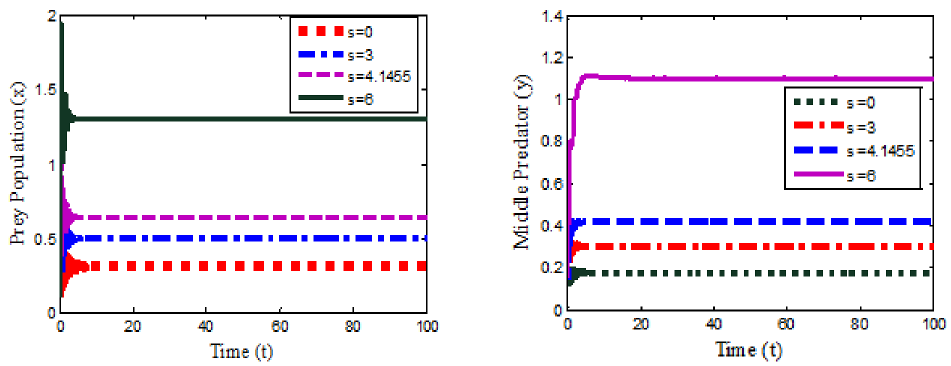

Fig. 2(a-b). Varation of prey and middle preator population with time for different tax levels.

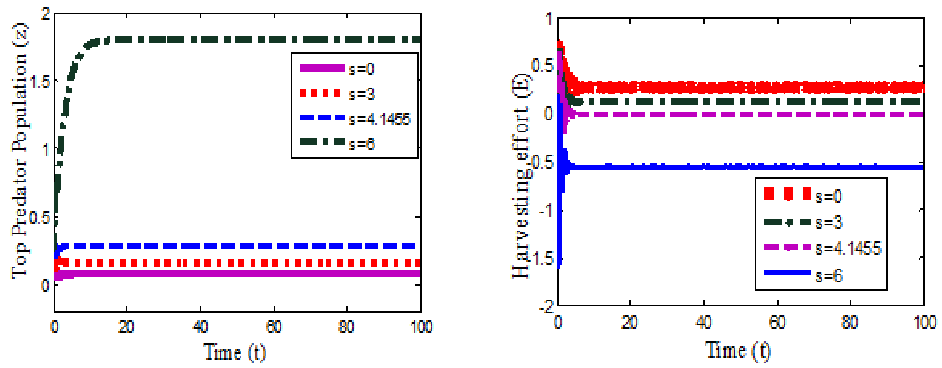

Fig. 2(c-d). Varation of top predator population and harvesting effort with time for different tax levels.

8. Conclusion

In this paper, we have considered and analyzed a ratio dependent food chain model with Michalies-Menten type functional response, where all the species are subjected to combined harvesting effort with tax as a control instrument to avoid over exploitation of populations. The above situation is described by means of a system with four non-linear differential equations.

The system is analyzed for boundedness of solutions, which is in turn, implies that system is biologically well behaved. The existence conditions for equilibrium points of the system are determined and its local stability. The stability of the system implies that prey-middle predator and top predator population and harvesting effort settle down to their respective equilibrium level under certain conditions. Using Pontryagins Maximum principle, an optimal policy to harvest food chain population with ratiodependent functional form has been discussed and optimal equilibrium levels of prey-predator population, effort and tax have been obtained. It has been shown that the total users cost of harvest per unit effort is equal to the present value of marginal revenue of effort at the optimal equilibrium level. It has also been noted that increase in discount rate decreases the economic rent and even it may tend to zero if the discount rate tends to infinity. All our important mathematical findings and graphical representation of variety of solutions of system (1) are depicted by using MATLAB programming.

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References

[1] **Clark, C. W.:** Bioeconomic Modeling and Fisheries Management, John Wiley, New York, 1985.

[2] **Mesterton-Gibbons, M.:** On the optimal policy for combined harvesting of predator-prey,Nat. Res. Model, 3 (1988), 63-90.

[3] **Clark, C. W.:** Mathematical Bioeconomics, the optimal Management of Renewable Resources, John Wiley, New York, 1990.

[4] **Azar, C., Holmberg, J. and Lindgren, K.:** Stability analysis of harvesting in predator prey model, J. of Theoretical Biology, 174 (1995), 13-19.

[5] **Srinivasu, P. D. N., Ismail, S. and Naidu, C. R.:** Global dynamics and controllability of a harvested prey-predator system, Journal of Biological System, 9(1) (2001), 67-79.

[6] **Armsworth, P. R. and Roughgarden, J. F.:** An invitation to ecological economics, Trends Ecol. Evol., 16 (2001), 229-234.

[7] **Kumar, S., Srivastava, S. K. and Chingakham, P.:** Hopf bifurcation and stability analysis in a harvested one-predator-two prey model, Applied Mathematics and Computation, 129 (2002), 107-118.

[8] **Hsu, S. B., Huang, T. W. and Kuang, Y.:** A ratio dependent food chain model and its applications to biological control, Math. Biosci., 181 (2003), 5583.

[9] **Moghadas, S. M. and Gummel, A. B.:** Dynamical and numerical analyses of a generalizedfood chain model, Applied math. and comp., 142(I) (2003), 35-49.

[10] **Kar, T. K., Misra, S. K. and Mukhopadhyay, B.:** A Bioeconomic model of a ratio-dependent prey- predator system and optimal harvesting, J. Appl. Math. and Computing, 22(1-2) (2006), 387- 401.

[11] **Lenard, B., Lewis, C. and Rebaza, J.:** Dynamics of ratio dependent predator prey models with non-constant harvesting, Disc. Cont. Dyn. Syst. S, 1(2) (2008), 303-315.

[12] **Kar, T. K. and Batabayal, A.:** Persistence and stability of a two prey one predator system, Int. J. Eng., Sci. and Tech., 2(2) (2010), 174-190.

[13] **Lenzini, P. and Rebaza, J.:** Non-constant predator harvesting on ratio-dependent predator-prey models, Applied Mathematical Sciences, 4(16) (2010), 791-803.

[14] **Agarwal, M. and Singh, V.:** Persistence and stability for three species ratio-dependent food chain model, International Journal of Engineering, Science and Tecnology, 3, No. 1 (2011), 305-322.

[15] **Agarwal, M. and Singh, V.:** Rich Dynamics of a food chain model with ratio-dependent type III functional response, International Journal of Engineering, Science and Technology, 5, No. 3 (2013), 124-141.