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Certain Expansions Involving Generalized Hypergeometric Series

By

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Abstract

The aim of this paper is to evaluate multiple series summation involving H function.

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1. Introduction

Further establish certain triple summation of hypergeometric function in terms of another hypergeometric function. Here we have made an attempt to show how this method can be utilized to establish multiple series summation involving H-function in terms of another H-function.

2. Notations, Definitions and well known Results

The H-function of one variable is defined as [4]

$$H_{p,q}^{m,n}\left[\begin{array}{c|c} x & \begin{array}{c}(\alpha_p;a_p)\\ \\ (\beta_q;b_q)\end{array}\end{array}\right]$$
$$=\frac{1}{2\pi i}\int_L \frac{\Gamma\left[1-(\alpha_n)+(a_n)\xi\right].\Gamma\left[(\beta_m)-(b_m)\xi\right]x^{\xi}d\xi}{\Gamma\left[(\alpha_{n+1,p})-(a_{n+1,p})\xi\right]\ \left[1-(\beta_{m+1,q})+(b_{m+1,q})\xi\right]} \tag{2.1}$$

where L is the usual notation for Barne's type contour integral. The following assumptions are made.

(a) $a_i > 0\ ; i = 1, 2, ..., p$
$b_j > 0\ ; i = 1, 2, ..., q$
(b) All the poles of integrand are simple.
(c) $O \leq m \leq q$ and $O \leq n \leq p$.
(d) The sequence of parameters $[(\beta_m); (b_m)]$ and $[(\alpha_n); (a_n)]$ are such that the none of the poles coincide. The path of integration is indented, if necessary in such a way that poles of $\Gamma[(\beta_m) - (b_m)\xi]$ lie on the right and those of $\Gamma[1 - (\alpha_n) + (a_n)\xi]$ lie on the left of the imaginary axis.

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(e) The integral (2.1) converges under certain conditions which can be easily obtained by considering the behaviour of the integrand one the closed contour.

We consider the integral in the plane taken round the closed contour L consisting of the imaginary axis from $-iR$ to $+iR$ and that part of semi circle $|\xi| = R$ which lies to the right of the imaginary axis, R being so chosen that the circle always passes between the poles of the integrand.

The following notations and known results have been used to evaluate certain expansions,

$$(\alpha_n) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \tag{2.2}$$

$$(\alpha_n) = \frac{(-)^n.\Gamma(1-\alpha)}{\Gamma(1-\alpha-n)} \tag{2.3}$$

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}.\Gamma(z).\Gamma(z+1/2) \tag{2.4}$$

$$\sum_{r=0}^{m}\sum_{s=0}^{n} \frac{(-m)_r(-n)_s(a)_s(c-a)_r(b)_s(c-b)_r}{(1)_r(1)_s(c)_{r+s}(1-a-b+c-m)_r(1+a+b-c-n)_s}$$
$$= \frac{(a)_m(b)_m(c-a)_n(c-b)_n}{(c)_{m+n}(a+b-c)_m(c-a-b)_n} \tag{2.5}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m(a')_n(b)_{m+n}(c)_m(c')_n}{(1)_m(1)_n(b)_m(b)_n(c+c')_{m+n}}$$
$$= \Gamma\left[\begin{array}{c} c-a', c'-a, b, b-a-a', c+c, \\ \\ c+c'-a-a', b-a, b-a', c, c' \end{array}\right] \tag{2.6}$$

where

$Rl[c+c'-b] \geq 0$,
$Rl[b-a-c] > 0$,
$Rl[b-a'-c'] > 0$.

$$(\alpha)_{M+N} = (\alpha)_M.(\alpha+m)_N. \tag{2.7}$$

3. Summations

Here we shall establish two general summations, 3.1. **Summation-I.**

$$\sum_{u,v}\sum_{r,s}\frac{(d)_{u+v}(a)_{u+s}(c-a)_{v+r}(c-b)_{u+r}(-m)_r(-n)_s(b)_{v+s}}{(1)_u(1)_v(1)_r(1)_s(d)_u(d)_v(1+a+b-c-n)_s(2c+2n-a-b)_{u+v}}\times$$
$$\times\frac{(c+m+n)_{u+v}}{(c)_{u+v+r+s}(1-a-b+c-m)_r}$$
$$=\frac{(a)_m(b)_m(c-a)_n(c-b)_n}{(a+b-c)_m(c-a-b)_n(c)_{m+n}}\Gamma\left[\begin{array}{l}c-a-b+n-m,\\ \\ 2(c-a-b+n-m),\end{array}\right.$$
$$\left.\begin{array}{l}c-a-b+n-m,d,d-a-b-2m,2c+2n-a-b\\ \\ d-a-m,d-b-m,c-a+n,c-b+n\end{array}\right] \tag{3.1}$$

Proof. The left side of (3.1), can be put in the form,

$$\sum_{u,v}\frac{(d)_{u+v}(a)_u(c-a)_v(c-b)_u(b)_v(c+m+n)_{u+v}}{(1)_u(1)_v(d)_u(d)_v(2c+2n-a-b)_{u+v}(c)_{u+v}}\times$$
$$\times\sum_{r,s}\frac{(d)_{u+v}(a)_u(c-a)_v(c-b)_u(b)_v(c+m+n)_{u+v}}{(1)_r(1)_s(1+a+b-c-n)_s(c+u+v)_{r+s}(1-a-b+c-m)_r}.$$

Now, using (2.4) we get

$$\sum_{u,v}\frac{(d)_{u+v}(a)_u(c-a)_v(c-b)_u(b)_v(c+m+n)_{u+v}}{(1)_u(1)_v(d)_u(d)_v(2c+2n-a-b)_{u+v}(c)_{u+v}}\times$$
$$\times\frac{(a+u)_m(bv)_m(c-a+v)_n(c-b+u)_n}{(c+u+v)_{m+n}(a+b-c)_m(c-a-b)_n}.$$

Using (2.1) we get

$$\frac{(a)_m(b)_m(c-a)_n(c-b)_n}{(a+b-c)_m(c-a-b)_n(c)_{m+n}}\times$$
$$\times\sum_{u,v}\frac{(a+m)_u(b+m)_v(c-a+n)_v(c-b+n)_u(d)_{u+v}}{(1)_u(1)_v(d)_u(d)_v(2c+2n-a-b)_{u+v}}.$$

Now, making use of (2.5), we get

$$\frac{(a)_m(b)_m(c-a)_n(c-b)_n}{(a+b-c)_m(c-a-b)_n(c)_{m+n}}\Gamma\left[\begin{array}{l}c-a-b+n-m,\\ \\ 2(c-a-b+n-m),\end{array}\right.$$
$$\left.\begin{array}{l}c-a-b+n-m,d,d-a-b-2m,2c+2n-a-b\\ \\ d-a-m,d-b-m,c-a+n,c-b+n\end{array}\right]$$

which is required result.

3.2. **Summation-II.**

$$\sum_{u,v,r,s} \frac{(d)_{u+v}(a)_{u+s}(b)_{v+s}(c+m+n)_{u+v}(c-a)_{v+r}(c-b)_{u+r}(-m)_r(-n)_r}{(1)_u(1)_v(1)_r(1)_s(d)_u(d)_v(1+a+b-c-n)_s(2c+2n-a-b)_{u+v}} \times$$

$$\times \frac{1}{(c)_{u+v+r+s}(1-a-b+c-m)_r} \times$$

$$\times F \left[\begin{array}{l} a, d+u+v, a+u+s, 1+a+b-c-n, b+v+s, c+m+n+u+v, \\ \\ d+u, d+v, b, a, c+m+n, 1+a+b-c-n, a+b-c+m, \end{array} \right. \left. \begin{array}{l} c, a+b-c+m-r; e; x \\ \\ c+u+v+r+s, f \end{array} \right]$$

$$= \frac{(a)_m(b)_m(c-a)_n(c-b)_n}{(a+b-c)_m(c)_{m+n}} \frac{\{\Gamma(c-a-b+n-m)\Gamma(c-a-b+n-m)}{\{\Gamma(d-a-b)\Gamma(d-b-m)\Gamma(c-a+n)}$$

$$\frac{\Gamma(d)\Gamma(d-a-b-2m)\Gamma(2c+2n-a-b)\}}{\Gamma(c-b+n)\Gamma(2c-2a-2b+2n-2m)\}} \times$$

$$\times \Gamma \left[\begin{array}{l} a+m, b+m, c, a+b-c, 1-c+a+b-n, d, \\ \\ a, b, a+b-c+m, c+m+n, 1+c+a+b \end{array} \right. \left. \begin{array}{l} \frac{1}{2}-c+a+b-n+m; e; -4x \\ \\ 1-c+a+b+2m, \frac{3}{2}-c+a+b-n+m, f \end{array} \right]. \tag{3.2}$$

Proof. Making use of the result (3.1), we established the result (3.2), if we put $d = d+p$, $a = a+p$ and $b = c+p$, $c = c+p$ in the result (3.1), we get

$$\sum_{u,v,r,s} \frac{(d+p)_{u+v}(a+p)_{u+s}(c-a)_{v+r}(c-b)_{u+r}}{(1)_u(1)_v(1)_r(1)_s(d+p)_u(d+p)_v(1+a+b-c-n+p)_s} \times$$

$$\times \frac{(-m)_r(-n)_s(b+p)_{v+s}(c+m+n+p)_{u+v}}{(2c+2n-a-b)_{u+v}(c+p)_{u+v+r+s}(1-a-b+c-m-p)_r}$$

$$= \sum \frac{(a+p)_m(b+p)_m(c-a)_n(c-b)_n}{(a+b-c+p)_m(c-a-b-p)_n(c+p)_{m+n}} \times$$

$$\times \Gamma \left[\begin{array}{l} c-a-b+n-m-p, c-a-b+n-m-p, d+p, \\ \\ 2(c-a-b+n-m-p), d-a-m, \end{array} \right. \left. \begin{array}{l} d-a-b-2m-p, 2c+2n-a-b; \\ \\ d-b-m, c-a+n, c-b+n \end{array} \right].$$

Multiplying by $\sum \dfrac{(e)_p x^p}{(1)_p (f)_p}$ on both side, we get

$$\sum_{u,v,r,s} \frac{(d+p)_{u+v}(a+p)_{u+s}(c-a)_{v+r}(c-b)_{u+r}}{(1)_u(1)_v(1)_r(1)_s(d+p)_u(d+p)_v(1+a+b-c-n+p)_s} \times$$

$$\times \frac{(-m)_r(-n)_s(b+p)_{v+s}(c+m+n+p)_{u+v}}{(2c+2n-a-b)_{u+v}(c+p)_{u+v+r+s}(1-a-b+c-m-p)_r} . \frac{(e)_p x^p}{(1)_p (f)_p}$$

$$= \sum \frac{(e)_p x^p}{(1)_p (f)_p} . \frac{(a+p)_m (b+p)_m (c-a)_n (c-b)_n}{(a+b-c+p)_m (c-a-b-p)_n (c+p)_{m+n}} \times$$

$$\times \Gamma \left[\begin{array}{l} c-a-b+n-m-p, c-a-b+n-m-p, d+p, \\ \\ 2(c-a-b+n-m-p), d-a-m, \end{array} \right.$$

$$\left. \begin{array}{l} d-a-b-2m-p, 2c+2n-a-b; \\ \\ d-b-m, c-a+n, c-b+n \end{array} \right]$$

Now, using (2.6) and (2.1), we get

$$\sum_{u,v,r,s} \frac{(d)_{u+v}(a)_{u+s}(c-a)_{v+r}(c-b)_{u+r}(c+m+n)_{u+v}(-m)_r(-n)_s(b)_{v+s}}{(1)_u(1)_v(1)_r(1)_s(d)_u(d)_v(1+a+b-c-n)_s(2c+2n-a-b)_{u+v}} \times$$

$$\times \frac{1}{(c)_{u+v+r+s}(1-a-b+c-m-p)_r} \times$$

$$\times \sum_p \frac{(d)}{(l)_p} . \frac{(d+u+v)_p (a+u+s)_p (1+a+b-c-n)_p (b+v+s)_p}{(d+u)_p (d+v)_p (b)_p (a)_p (c+m+n)_p} \times$$

$$\times \frac{(c+m+n+u+v)_p (c)_p (a+b-c+m-r)_p (e)_p x^p}{(1+a+b-c-n+s)_p (a+b-c+m)_p (c+n+v+r+s)_p (f)_p}$$

$$= \frac{(a)_m (b)_m (c-a)_n (c-b)_n}{(a+b-c)_m (c)_{m+n}} \frac{\{\Gamma(c-a-b+n-m)\Gamma(c-a-b+n-m)}{\{\Gamma(d-a-m)\Gamma(d-b-m)\Gamma(c-a+n)}$$

$$\frac{\Gamma(d)\Gamma(d-a-b-2m)\Gamma(2c+2n-a-b)\}}{\Gamma(c-b+n)\Gamma(2c-2a-2b+2n-2m)\}} \times$$

$$\times \sum_p \frac{(a+m)_p (b+m)_p (c)_p (a+b-c)_p (1-c+a+b-n)_p (d)_p}{(1)_p (a)_p (b)_p (a+b-c+m)_p (c+m+n)_p (1-c+a+b)_p} \times$$

$$\times \frac{\left(\frac{1}{2}-c+a+b-n+m\right)_p (e)_p (-4x)^p}{(1-c+a+b+2m)_p \left(\frac{3}{2}-c+a+b-n+m\right)_m} . \frac{1}{(f)_p} .$$

Now, we get after summing the inner series

$$\sum_{u,v,r,s} \frac{(d)_{u+v}(a)_{u+s}(b)_{v+s}(c+m+n)_{u+v}(c-a)_{v+r}(c-b)_{u+r}(-m)_r(-n)_s}{(1)_u(1)_v(1)_r(1)_s(d)_u(d)_v(1+a+b-c-n)_s(2c+2n-a-b)_{u+v}} \times$$

$$\times \frac{1}{(c)_{u+v+r+s}(1-a-b+c-m-)_r}\left[\begin{array}{l} d, d+u+v, a+u+s, 1+a+b-c-n, \\ \\ (d+u), d+v, b, a, c+m+n, \end{array}\right.$$

$$\left.\begin{array}{l} b+v+s, c+m+n+u+v, c, a+b-c+m-r; e; x \\ \\ 1+a+b-c-n+s, a+b-c+m, c+u+v+r+s, f \end{array}\right]$$

$$= \frac{(a)_m(b)_m(c-a)_n(c-b)_n}{(a+b-c)_m(c)_{m+n}} \frac{\{\Gamma(c-a-b+n-m)\Gamma(c-a-b+n-m)}{\{\Gamma(d-a-b)\Gamma(d-b-m)\Gamma(c-a+n)}$$

$$\frac{\Gamma(d)\Gamma(d-a-b-2m)\Gamma(2c+2n-a-b)\}}{\Gamma(c-b+n)\Gamma(2c-2a-2b+2n-2m)\}} \times$$

$$\times \Gamma\left[\begin{array}{l} a+m, b+m, c, a+b-c, 1-c+a+b-n, d, \\ \\ a, b, a+b-c+m, c+m+n, 1+c+a+b \end{array}\right.$$

$$\left.\begin{array}{l} \frac{1}{2}-c+a+b-n+m; e; -4x \\ \\ 1-c+a+b+2m, \frac{3}{2}-c+a+b-n+m, f \end{array}\right],$$

which is required result.

3.3. **Expansion-III.**

$$\sum\sum \frac{(-m)_r(-n)_s(c-a)_{v+r}(c-b)_{u+r}}{(1)_u(1)_v(1)_r(1)_s(2c+2n-a-b)_{u+v}} \times H^{3+\mu,5+N}_{8+p,8+q}$$

$$\left[x \left| \begin{array}{l} (1-a-u-s;1), (1-b-v-s;1), (1-c-m-n-u-v;1), \\ \\ (d+u+v;1), (d,1), (1-a-b+c-m;1), (dq;Bq), (1-a;1), \end{array}\right.\right.$$

$$\begin{array}{l} (c-a-b+n;1), (1-c;1), (ap;Ap), (d+u;1), (d+v;1), \\ \\ (1-b;1), (1-c-m-n;1)(c-a-b+u-s;1), \end{array}$$

$$\left.\begin{array}{l} (1-a-b+c-m+r;1); \\ \\ (1-c-u-v-r-s;1) \end{array}\right]$$

$$= \frac{\Gamma(2c+2n-a-b)}{\Gamma(c-a)\Gamma(c-b)\Gamma(d-a-m)\Gamma(d-b-m)} \times$$

$$\times H^{3+\mu,5+N}_{8+p,8+q}\left[x \left| \begin{array}{l} (1-a-u-s;1), (1-b-v-s;1), \\ \\ (d+u+v;1), (d,1), (1-a-b+c-m;1), \end{array}\right.\right.$$

$$(1-c-m-n-u-v;1),(c-a-b+n;1),(1-c;1),(ap;Ap),(d+u;1),$$

$$\left.\begin{matrix}(dq;Bq),(1-a;1),(1-b;1),(1-c-m-n;1) \\ (d+v;1),(1-a-b+c-m+r;1); \\ (c-a-b+u-s;1),(1-c-u-v-r-s;1)\end{matrix}\right] \tag{3.3}$$

where

(a) $0 \leq \mu q, 0 \leq N \leq p,$

(b) $$\left[\begin{array}{l} \sum\limits_{i=1}^{p} A_i - \sum\limits_{i=1}^{q} B_i - 1 < 0, \\ |\arg x| < \pi \left[\sum\limits_{i=1}^{N} A_i - \sum\limits_{i=1+\mu}^{q} B_i\right] \\ \text{or} \\ \sum\limits_{i=1}^{p} A_i - \sum\limits_{i=1}^{q} B_i - 1 = 0, \\ |\arg x| < \pi \left[\sum\limits_{i=1}^{N} A_i - \sum\limits_{i=1+\mu}^{q} B_i\right] \\ \text{and } |x| < 1 \end{array}\right. .$$

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References

[1] **Agarwal Praveen, Qi Feng, Mehar Chand, Jain, Shilpi:** Certain integrals involving the generalized hypergeometric function and the Laguerre polynomials, Journal of Computational and Applied Mathematics, 313(4) (2016), 307-317.

[2] **Denis, R. Y.:** On certain expansions of generalized hypergeometric functions involving free parameters, Rendiconti del Circolo Matematico di Palermo, 21 (1972), 42-46.

[3] **Erdelyi, A. et al.:** Higher Transidental Function, Vol. I, McGraw Hill, New York, 1953.

[4] **Fox, Charles:** The G and H-Functions as Gymmetrical Fourier-kernals, Trans. Amer, Math. Soc., 98 (1961), 395-429.

[5] **Gaboury Sebastien and Tremblay Richard:** An Expansion theorem involving H-function of several complex variables, Hindawi Publishing Corporation, International Journal of Analysis, Vol. 2013, Article ID 353547, 7 pages, http://dx.doi.org/10.1155/2013/353547.

[6] **Ragab, F. M.:** Proceedings, Glassgow-math Assoc. 1, (1953), 192-194.

[7] **Rathie, Arjun K.:** A new Generalization of Generalized Hypergeometric Functions, Le Matematiche, LII (1997) Fasc. II, 297-310.

[8] **Saxena, R. K.:** Some Theorems on Generalized Laplace Transform -I, Proc. nat. Inst. Sc. (India), 26 (1960), 400-413, Part -A.

[9] **Verma, A.:** A note on the evaluation of certain integrals involving G -function, Ganita 16 (1965), 51-54.

[10] **Mathai, A. M. and Saxena, R. K.:** The H-functions with applications in statics and other disciplines, John Wiley & Sons Inc, New York, 1978.