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Study On Special Weakly Symmetric Manifolds

By

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Abstract

Symmetric spaces and weakly symmetric spaces had studied by a large number of authors such as Chaki and Gupta [1], Desai and Amur [2] Tamassay and Binh [11] and Singh and Khan [8] etc. Recently, Singh and Khan [9] introduced and studied the notion of a Special Weakly Symmetric Riemannian manifolds and denoted such manifold by $(SWS)_n$. In this paper, we have studied some properties of a special weekly specially symmetric Riemannian manifold $(SWSS)_n$ and have investigated some interesting and fruitful results on it.

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1. Introduction

Let (M^n, g) be a Riemannian manifold of dimension n with a Riemannian metric g and $\chi(\mathrm{M})$ denote the set of differentiable vector fields on M^n. Let $K(X, Y, Z)$ be the Riemannian curvature tensor of type (1,3) for $X, Y, Z \in \chi(\mathrm{M})$. A non - flat Riemannian manifold $(\mathrm{M}^n,\ g)$, $(n \geq 2)$ is called a special weakly symmetric Riemannian manifold [9], if its curvature tensor K of type (1, 3) satisfies the condition.

$$\begin{aligned}(\nabla_X K)(Y, Z, V) =& 2\alpha(X)K(Y, Z, V) + \alpha(Y)K(X, Z, V)\\ &+ \alpha(Z)K(Y, X, V) + \alpha(V)K(Y, Z, X)\end{aligned} \tag{1.1}$$

where α is a 1-form and is defined as

$$\alpha(X) = g(X, \rho), \tag{1.2}$$

for every vector field X and ∇ denotes the operator of covariant differentiation with respect to the metric g. Such a manifold is denoted by $(SWS)_n$. If we replace K by J in (1.1), then it reduces to

$$\begin{aligned}(\nabla_X J)(Y, Z, V) =& 2\alpha(X)J(Y, Z, V) + \alpha(Y)J(X, Z, V)\\ &+\alpha(Z)J(Y, X, V) + \alpha(V)J(Y, Z, X),\end{aligned} \tag{1.3}$$

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where J is the special curvature tensor defined by (see in [4][10])

$$J(X,Y,Z) = K(X,Y,Z) + K(X,Z,Y) \tag{1.4}$$

which satisfies the following properties.

$$J(X,Y,Z) = J(X,Z,Y) \tag{1.5}$$

and

$$J(X,Y,Z) + J(Y,Z,X) + J(Z,X,Y) = 0. \tag{1.6}$$

Such an n-dimensional Riemannian manifold shall be called a special weakly specially symmetric Riemannian manifold and such a manifold is denoted by $(SWSS)_n$.
A Riemannian manifold is recurrent [10] if

$$(\nabla_X K)(Y,Z,V) = \alpha(X)K(Y,Z,V) \tag{1.7}$$

where $\alpha(X)$ is a recurrent parameter.

The above results will be used in the next section.

2. Existence of a (SWSS)$_n$

Let(M^n, g) be a (SWS)$_n$. Taking covariant derivative of (1.4) with respect to X and then using (1.1), we get

$$\begin{aligned}(\nabla_X J)(Y,Z,V) =& 2\alpha(X)K(Y,Z,V) + \alpha(Y)K(X,Z,V) + \alpha(Z)K(Y,X,V)\\ &+ \alpha(V)K(Y,Z,X) + 2\alpha(X)K(Y,V,Z)\\ &+ \alpha(Y)K(X,V,Z) + \alpha(V)K(Y,X,Z) + \alpha(Z)K(Y,V,X)\\ =& 2\alpha(X)[K(Y,Z,V) + K(Y,V,Z)] + \alpha(Y)[K(X,Z,V)\\ &+ K(X,V,Z)] + \alpha(Z)[K(Y,X,V) + K(Y,V,X)]\\ &+ \alpha(V)[K(Y,Z,X) + K(Y,X,Z)].\end{aligned} \tag{2.1}$$

Using (1.4) in (2.1), we have

$$\begin{aligned}(\nabla_X J)(Y,Z,V) =& 2\alpha(X)J(Y,Z,V) + \alpha(Y)J(X,Z,V)\\ &+\alpha(Z)J(Y,X,Z) + \alpha(V)J(Y,Z,X)\end{aligned}$$

This leads us to the following:

Theorem 1. A $(SWS)_n$, is necessarily $(SWSS)_n$.

Let (M^n, g) be $(SWSS)_n$. Taking covariant derivative of (1.4) with respect to X and then using (1.3), we have

$$2\alpha(X)J(Y,Z,V)+\alpha(Y)J(X,Z,V)+\alpha(Z)J(Y,X,V)+\alpha(V)J(Y,Z,X) = (\nabla_X K)(Y,Z,V)+(\nabla_X K)(Y,V,Z)$$

which in view of (1.4) gives

$$\begin{aligned}(\nabla_X K)(Y,Z,V)+(\nabla_X K)(Y,V,Z) = {} & 2\alpha(X)[K(Y,Z,V)+K(Y,V,Z)]\\ & +\alpha(Y)[K(X,Z,V)+K(X,V,Z)]\\ & +\alpha(Z)[K(Y,X,V)+K(Y,V,X)]\\ & +\alpha(V)[K(Y,Z,X)+K(Y,X,Z)]\end{aligned} \tag{2.2}$$

Permuting equation (2.2) twice with respect to X, Y, Z; adding the three obtained equations and then using Bianchi's first and second identities and; Skew symmetric property of curvature tensor, we have

$$\begin{aligned}(\nabla_X K)(Y,V,Z)+(\nabla_Y K)(Z,V,X)+(\nabla_Z K)(X,V,Y) & \\ =2[\alpha(X)\{K(Y,V,Z)+K(Z,V,Y)\} & \\ +\alpha(Y)\{K(Z,V,X)+K(X,V,Z)\} & \\ +\alpha(Z)\{K(X,V,Y)+K(Y,V,X)\}] &\end{aligned} \tag{2.3}$$

which in view of (1.7) gives

$$\begin{aligned}2\alpha(X)K(Y,V,Z)+\alpha(Y)K(Z,V,X)+\alpha(Z)K(X,V,Y) & \\ =2[\alpha(X)K(Z,V,Y)+\alpha(Y)K(X,V,Z)+\alpha(Z)K(Y,V,X)]. &\end{aligned} \tag{2.4}$$

Contracting (2.4) with respect to X and using the fact $(C_1^1 K)$

$Ric(Y,Z)$, $(C_3^1 K)=0$ and

$$\alpha(K(Y,V,Z)) = \acute{K}(Y,V,Z,\rho),$$

we have

$$\acute{K}(Y,V,Z,\rho)+\alpha(Z)Ric(V,Y) = 2[\acute{K}(Z,V,Y,\rho)+\alpha(Y)Ric(V,Z)]$$

which in view of

$$\acute{K}(X,Y,Z,V)+\acute{K}(Z,V,X,Y) \qquad \text{and} \qquad Ric(X,Y)=Ric(Y,X)$$

gives

$$\acute{K}(Z,\rho,Y,V)+\alpha(Z)Ric(Y,V) = 2[\acute{K}(Y,\rho,Z,V)+\alpha(Y)Ric(Z,V)] \tag{2.5}$$

Factoring off V in (2.5), we have

$$K(Z,\rho,Y)+\alpha(Z)R(Y) = 2[K(Y,\rho,Z)+\alpha(Y)R(Z)]$$

which on contraction with respect to Z gives

$$Ric(\rho, Y) + Ric(Y, \rho) = 2\alpha(Y)r$$

which in view of $Ric(X, Y) = Ric(Y, X)$ gives

$$Ric(Y, \rho) = \alpha(Y)r$$

which in view of g(X,ρ) = α(X) gives

$$\alpha(R, Y) = \alpha(Y)r$$

Thus, we have the following result.

Theorem 2. In $(SWSS)_n$, the scalar curvature r is related as

$$\alpha(Y)r = \alpha(R(Y)) \quad \text{for recurrent manifold.}$$

Taking Covariant derivative of (1.4) with respect to X, we have

$$(\nabla_X J)(Y, Z, V) = (\nabla_X K)(Y, Z, V) + (\nabla_X K)(Y, V, Z)$$

Taking Cyclic sum of the above relation and using Bianchi's second identity, we have

$$\begin{aligned} &(\nabla_X J)(Y, Z, V) + (\nabla_Y J)(Z, X, V) + (\nabla_Z J)(X, Y, V) \\ &= (\nabla_X K)(Y, V, Z) + (\nabla_Y K)(Z, V, X) + (\nabla_Z K)(X, V, Y) \end{aligned} \tag{2.6}$$

Let (M^n,g) is $(SWSS)_n$. Then in view of (1.3) and (1.4), The relation (2.6) reduces to

$$\begin{aligned} &2\alpha(X)[K(Y, Z, V) + K(Y, V, Z)] + \alpha(Y)[K(X, Z, V) + K(X, V, Z)] \\ &+ \alpha(Z)[K(Y, X, V) + K(Y, V, X)] + \alpha(V)[K(Y, Z, X) + K(Y, X, Z)] \\ &+ 2\alpha(Y)[K(Z, X, V) + K(Z, V, X)] + \alpha(Z)[K(Y, X, V) + K(Y, V, X)] \\ &+ \alpha(X)[K(Z, Y, V) + K(Z, V, Y)] + \alpha(V)[K(Z, X, Y) + K(Z, Y, X)] \\ &+ 2\alpha(Z)[K(X, Y, V) + K(X, V, Y)] + \alpha(X)[K(Z, Y, V) + K(Z, V, Y)] \\ &+ \alpha(Y)[K(X, Z, V) + K(X, V, Z)] + \alpha(V)[K(X, Y, Z) + K(X, Z, Y)] \\ &= (\nabla_X K)(Y, V, Z) + (\nabla_Y K)(Z, V, X) + (\nabla_Z K)(X, V, Y). \end{aligned} \tag{2.7}$$

Using (1.7) in (2.7) and using Bianchi's first identity and skew symmetric property of K(X,Y,Z), we have

$$\begin{aligned} \alpha(X)[K(Y, V, Z) + 2K(Z, V, Y)] + &\alpha(Y)[K(Z, V, X) + 2K(X, V, Z)] \\ &+\alpha(Z)[K(X, V, Y) + 2K(Y, V, X)] = 0. \end{aligned} \tag{2.8}$$

Contracting (2.8) with respect to X and using (C_1^1K) = Ric(Y,Z), (C_3^1K)=0 and

$$\alpha(K(Y, Z, V)) = \acute{K}(Y, Z, V, \rho),$$

we have

$$\acute{K}(Y,V,Z,\rho) + 2\acute{K}(Z,V,Y,\rho) + 2\alpha(Y)Ric(V,Z) + \alpha(Z)Ric(V,Y) = 0$$

which in view of

$$\acute{K}(X,Y,Z,V) = \acute{K}(Z,V,X,Y)$$

gives

$$\acute{K}(Z,\rho,Y,V) + 2\acute{K}(Y,\rho,Z,V) + 2\alpha(Y)g(R(Z),V) + \alpha(Z)g(R(Y),V) = 0 \quad (2.9)$$

Factoring off V in (2.9), we have

$$K(Z,\rho,Y) + 2K(Y,\rho,Z) + 2\alpha(Y)R(Z) + \alpha(Z)R(Y) = 0$$

which on contracting with respect to Z gives

$$\begin{aligned} Ric(\rho,Y) + 2\alpha(Y)r + Ric(Y,\rho) &= 0 \\ \text{or} \quad Ric(Y,\rho) + \alpha(Y)r &= 0 \end{aligned} \quad (2.10)$$

This leads us to the following result.

Theorem 3. In $(SWSS)_n$, the scalar curvature r and Ricci tensor of type (0,2) is related as (2.10), for recurrent manifold.

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