J. Nat. Acad. Math.
Vol. 34 (2020), pp.36-44

ISSN 0970-5228

On Generalized Product of Cesáro Summable Series

By

Suyash Narayan Mishra

Abstract

Summability is more general than that of ordinary convergence. This is a branch of mathematical analysis in which an infinite series which is usually divergent can converge to a finite sum s (say) by ordinary summation techniques and become summable with the help of different summation means or methods. Many authors have discussed various summability methods. C method was given by Ernesto Cesáro such that ordinary Cesáro summation was written as $(C, 1)$ summation whereas generalised Cesáro summation was given as (C, α). In 1913, Hardy [5] proved a Theorem on (C, α), $\alpha > 0$ summability of the series. (C, α), $[C, \alpha]$, $|C, \alpha|$ denotes respectively ordinary, strong and absolute Cesáro summability methods. The product of ordinary and absolute summability has been discussed by Borwein [1]. In this paper generalized product of ordinary and absolute summability has been defined and some of its properties investigated.

Keywords and Phrases: (C, α), $[C, \alpha]$, $|C, \alpha|$ means, (C, α, β) means, Lebesgue Integral.
AMS Subject Classification: 42B05,42B08.

1. Introduction

Borwein [2] introduced the ordinary summability method (C, α) for series and investigated some of its properties. (C, α), $[C, \alpha]$, $|C, \alpha|$ denotes respectively ordinary, strong and absolute Cesáro summability methods. The method $[C, \alpha]$, previsously defined only for $\alpha \geq 0$, is defined in a natural way for $\alpha < 0$. It is known that if $\sum_0^\infty a_n$ is summable $|C, -\alpha, \beta|$ to A and $\sum_0^\infty b_n$ is summable $(C, -\alpha, \beta)$ to B, where $\alpha \geq 0$, then $\sum_0^\infty c_n$ is summable $(C, -\alpha, \beta)$ to AB. In [2], the product of ordinary and absolute summability has been given and some of its properties has been discussed. Mishra and Srivastava [8] introduced the Summability method (C, α, β) for functions by generalizing (C, α) summability method. Mishra and Mishra [9] discussed Strong Summability of functions based

Received : May 15, 2020; Accepted : November 2, 2020

on $(D,k)(C,\alpha,\beta)$ Summability methods. In this paper, we discuss generalized product of ordinary and absolute summability and investigate a relation between different sets of parameters. The case $\beta = 0$ is due to [2].

Throughout this paper $\sum_0^\infty c_n$ denotes the Cauchy product of the series $\sum_0^\infty a_n$ and $\sum_0^\infty b_n$, i.e.

$$c_n = \sum_{r=0}^{n} a_r b_{n-r}. \tag{1.1}$$

Now we can prove the following Theorems based on above results.

Theorem 1. If $\alpha \geq 0$ and $\sum_0^\infty a_n$, $\sum_0^\infty b_n$ are summable $[C,-\alpha,\beta]$ to $A.B$. The case $\beta = 0$ of this Theorem has been established by Boyd [4]. We also prove the following two Theorems.

Theorem 2. There are series $\sum_0^\infty a_n$, $\sum_0^\infty b_n$ respectively summable $[C,-I,\beta]$ and absolutely convergent, for which $\sum_0^\infty c_n$ is not summable $[C,0,\beta]$.

Theorem 3. Given $\alpha \geq -1$, there are series $\sum_0^\infty a_n$, $\sum_0^\infty b_n$ respectively summable $[C,-I,\beta]$ and (C,α,β), for which $\sum_0^\infty c_n$ is not summable $[C,\alpha+1,\beta]$.

The cases $\alpha = -1$ and $\alpha = 0$ of Theorem 3 have been proved by Boyd [4]. The case $\beta = 0$ is due to [1].

2. Notation, Definitions and Preliminary Result

Let

$$s_n = \sum_{r=0}^{n} a_r \qquad (n = 0, 1, \ldots). \tag{2.1}$$

Given matrices $Q = (q_{n,r})$, $P = (P_{n,r})$ $(n, r = 0, 1, ...)$ with $P_{n,r} \geq 0$, the strong summability method $[P,Q,R]$ is defined (see [3]) as follows. Let

$$\sigma_n = Q(S_n) = \sum_{r=0}^{\infty} q_{n,r} S_r. \tag{2.2}$$

Then $\sum_0^\infty a_n$ is summable $[P,Q,R]$ to s, and we write $S_n \to S[P,Q,R]$, if

$$\sum_{r=0}^{\infty} P_{n,r} |\sigma_r - s| \tag{2.3}$$

is defined for each n and tends to 0 and $n \to \infty$.

$$\epsilon_n^a = \binom{n+a}{n}, \triangle a s_n = \sum_{r=0}^{n} \epsilon_{n-r}^{-\alpha-1} s_r (n = 1, \ldots; \text{ any real } \alpha). \tag{2.4}$$

The (C, α, β) transform of $f(x)$, which we denote by $\partial_{\alpha,\beta}(x)$ is given by

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} \frac{1}{x^{\alpha+\beta}} \int_0^x (x-y)^{\alpha-1} y^{\beta} f(y) dy, \quad (\alpha > 0, \beta > -1), \tag{2.5}$$

If this exists for $x > 0$ and $\partial_{\alpha,\beta}(x)$ tends to a limit s as $x \to \infty$, we say that $f(x)$ is summable (C, α, β) to s, and we write $f(x) \to s(C, \alpha, \beta)$.

If $\alpha \geq 0$, $p \geq 1$, $\beta > -1$, we say that $f(y)$ is summable $|C, \alpha, \beta|_p$ (absolutely summable (C, α, β) with index p, if

$$\int_T^{\infty} y^{p-1} \left| \frac{d}{dy} \partial_{\alpha,\beta}(y) \right|^p dy < \infty. \tag{2.6}$$

Define $C_{\alpha,\beta}$ to be the matrix $C_{\alpha,0,\beta}$ when $\alpha > -1$, and $C_{\alpha,-\alpha}$ when $\alpha \leq -1$. Then, for any real α,β the statement $\sum_0^{\infty} \alpha_n$ is summable (C, α, β) to $A\beta$ can be interpreted (see [5]) as

$$\sigma_n = C_{\alpha,\beta}(s_n) \to A + \beta.$$

We now define, for every real α, β the strong Cesáro method $[C, \alpha, \beta]$ to be $[C_1, C_{\alpha-1}, \beta]$. The definition is standard for $\alpha, \beta > 0$; for $\alpha > 0$; the method $[C, \alpha, \beta]$ does not appear to have been defined explicity before. The following proposition, which is a special case of a known result ([3] with $X = C_{-1,1}$), shows that our definition of $[C, 0, \beta]$ is equivalent to one framed by Hyslop [6]. Following inclusion results are obvious.

I. The series $\sum_0^{\infty} \alpha_n$ is summable $[C, 0, \beta]$ to $A+\beta$ if and only if it is convergent with sum A and

$$\sum_{r=0}^{\infty} r|\alpha_r| = O(n + \beta).$$

Given summability methods X, Y we say that X is included in Y and write $X \subseteq Y$ if every series summable X is also Y to the same sum; X and Y are said to be equivalent and we write $X \simeq Y$ if each is included in the other.

We list next time inclusions, which hold for every real α, β together with reference to results of which they are immediate consequences.

II. $[C, \alpha, \beta] \simeq [C_1, C_{\alpha-1,\beta}]$ $(\beta > -1, \alpha + \beta > 0)$.
III. $[C, \alpha, \beta] \subseteq [C, \alpha + \delta, \beta][\delta, \beta > 0]$.
IV. $[C, \alpha - 1, \beta] \subseteq [C, \alpha, \beta] \subseteq [C, \alpha, \beta]$.
V. $|C, \alpha, \beta| \subseteq [C, \alpha, \beta]$.

As in the case with series, if an integral is (C, α) summable for some value of $\alpha \geq 0$, then it is also (C, β) summable for all $\beta > \alpha$, and the value of the resulting limit is the same [12].

3. Proof of the Theorems

In order to prove Theorem 1 we require a Lemma which is similar to one proved by Winn ([7]), 483-484).

Lemma 3.1. If $r, u, v > 0$ and α, β satisfying same conditions, we have

$$S_n = \sum_{r=0}^{n} s_r = O(n) \tag{3.1}$$

then, for $\alpha < 1$, $\sum\limits_{r=0}^{n} \epsilon_{r-u}^{-\alpha,\beta} s_{r+u} = O(\epsilon_n^{1-\alpha,\beta})$.

Proof. By partial integration over N, we have

$$\epsilon_{r-u}^{-\alpha,\beta} s_{r+u} = (r+1)[\int_1^N |\int_0^1 (1 - f(v))\{C_{\alpha,\beta}(vy_v) - r\}|C_{\alpha,\beta}(vy_{v-1})^{1/\alpha}]^{\alpha}, \tag{3.2}$$

Therefore

$$\epsilon_{r-u}^{-\alpha,\beta}\, s_{r+u} \to 0. \tag{3.3}$$

Since $\frac{w_r}{(r+1+\beta)} \to 0$ and $\sum\limits_{r=0}^{n} \epsilon_r^{-\alpha,\beta} = \epsilon_n^{1-\alpha,\beta}$, the required result can now be obtained by an application of Hyslop's Theorem.

Proof of Theorem 1.

Case (i). Suppose $A = B = 0$.

Let $\mu = m + \alpha + \beta$, where m is a non-negative integer and $0 \leq \alpha + \beta + r < 2$; and let

$$\alpha + r - \beta = \sum_{r=0}^{n} \epsilon_r^{-\alpha,} \frac{s_r}{r+1} + O(\epsilon_n^{1-\alpha}).$$

Hence

$$s_n = \sum_{r=0}^{n} \alpha_{r,\beta}\ t_n = \sum_{r=0}^{n} b_{r,\beta}. \tag{3.4}$$

It has been shown ([1], 47) that a necessary and sufficient condition for $\sum_0^\infty c_n$ to be summable $(C, -\mu, \beta)$ to 0 is that

$$X_{n+r} + Y_{n+r} + Z_{n+r} = O(1 + \beta + r), \tag{3.5}$$

where

$$X_{n+r} = \sum_{p=1}^{m} \frac{1}{\epsilon_{n+r}^{1-\alpha)}} \sum_{r=0}^{n} \triangle^{m+1-p+r}(\epsilon_r^{m+1-p} s_r) \tag{3.6}$$

when $m + r + v \geq 1$ and $X_n = 0$ when $m + \beta = 0$, and

$$Y_{n+r} = \frac{1}{\epsilon_{n+r}^{1-\alpha}} \sum_{r=0}^{n} t_{n-r} \triangle^{\mu+1-\beta}(\epsilon_r^{\mu+1+\beta})\left(\frac{\epsilon_n^{\mu+1-\alpha}}{s_{r+v}}\right), \tag{3.7}$$

$$Z_{n+r} = \frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} s_{n-r} \triangle^{\mu+1}(\epsilon_{r+n}^{\mu+1-\alpha} t_{r+v}), \tag{3.8}$$

By hypothesis, $s_n \to 0[C, -\mu, \beta]$, $t_n \to 0[C, -\mu, \beta]$, so that by the second inclusion in IV [8],

$$s_n \to 0[C, -\mu, \beta], \qquad t_n \to 0[C, -\mu, \beta]; \tag{3.9}$$

and a known consequence ([3], 47-49) is that

$$(C, \alpha) - \sum_{j=0}^{\infty} a_j = \lim_{n \to \infty} \sum_{j=0}^{n} \frac{\binom{n}{j}}{\binom{n+\alpha}{j}} a_j.$$

Now, let

$$y_n = \triangle^{\mu+1}(\epsilon_r^{\mu+1-\alpha} s_n) = \epsilon_r^{-\alpha} C_{-\mu-1,\mu+1-\alpha}(s_n) \tag{3.10}$$

from the hypothesis $s_n \to 0[C, -\mu, \beta]$ we deduce, by II, that

$$\sum_{r=0}^{n} \frac{|y_r|}{\epsilon_r^{-\alpha,\beta}} + O(n). \tag{3.11}$$

and hence, by the Lemma, that

$$\frac{1}{\epsilon_n^{1-\alpha,\beta}} \sum_{r=0}^{n} |y_r| = O(1). \tag{3.12}$$

Next, since $t_n \to 0[C, -\mu, \beta]$, we have by III, that $t_n = O(1)$ and it follows that

$$Y_n = \frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} t_{n-r} y_r = O(1). \tag{3.13}$$

Similarly $Z_n = O(1)$; and the proof of Case (i) is complete.

Case (ii). Suppose now that there are no restrictions on A, B.

Let $\alpha_0' = \alpha_0 - A$, $b_0\prime = b_0 - B$; $\alpha_r' = \alpha_r$, $b_r' = b_r (r > 0)$ and let

$$c_n' = \sum_{r=0}^{n} \alpha_r' b_{n-r}'. \tag{3.14}$$

Since $\sum_0^\infty \alpha_n$, $\sum_0^\infty b_n$ are summable $(C, -\mu)$ to A, B respectively, it is readily seen that $\sum_0^\infty \alpha_n$ and $\sum_0^\infty b_n$ are both summable $(C, -\mu, \beta)$ to 0, from which it follows, by Case (i), that $\sum_0^\infty \alpha_n'$ is summable $(C, -\mu)$ to 0. But

$$\sum_{n=0}^{\infty} A_n^\alpha x^n = \frac{\sum\limits_{n=0}^{\infty} a_n x^n}{(1-x)^{1+\alpha}}.$$

Hence

$$\sum_0^N \alpha_n = A \sum_0^N b_n - A + B, \tag{3.15}$$

and $\sum_0^\infty \alpha_n$, $\sum_0^\infty b_n$ are summable $(C, -\mu, \beta)$ to A, B respectively. Hence bounded $(C, \alpha, \beta)-$ variation over $(0, \infty)$ see [9].

Proof of Theorem 2. For convenience we divide the proof into three parts.

Case I. The case is obvious.

Case II. $\alpha = \beta + r$

We show now that given any unbounded sequence of positive number $\{U_n\}$, there is a sequence $\{u_n\}$ such that

$$U_n \geq 0, \quad \sum_0^\infty U_n < \infty \text{ and } \sum_0^N r v_r \neq 0(n). \tag{3.16}$$

Let $\{\beta_n\}$ be a sequence not converging to 0 such that

$$\beta_n \geq 0 \text{ and } \sum_0^\infty \frac{\beta_n}{U_n} < \infty; \tag{3.17}$$

a suitable sequence can be constructed by first defining an increasing sequence of positive n_v for which

$$U_{nv} > v^2,$$

and then taking β_n to be 1 whenever $n = n_v$ and 0 otherwise.

Let

$$\alpha_0 = 0,\ \alpha_n = \frac{\beta_n}{U_n} - \frac{n-1}{n}\frac{\beta_{n-1}}{U_{n-1}} \qquad (n \geq 1). \tag{3.18}$$

Then

$$U_n \sum_{0}^{n} r\alpha_r = n\beta_n$$

and

$$\sum_{0}^{\infty} |\alpha_n| < \infty.$$

Setting $u_n = |\alpha_n|$, we have

$$U_n \sum_{0}^{n} ru_r \geq n\beta_n, \tag{3.19}$$

and so the sequence $\{v_n\}$ satisfies (2) as required.

Case III. $n = \beta + r$

To prove our Theorem take $\alpha_n = (-1)^n u_n$ where $u_n > 0$, $nu_n = 0$ (1) and $\sum_0^\infty (-1)^n u_n$ is conditionally convergent; e.g. $u_n = \frac{1}{n+2}\log(n+2)$. Then $u_n = \sum_0^n u_r$ is positive and tends to infinity, and $\sum_0^\infty a_r$ is summable $(C-1, \beta)$. Let $b_n = (-1)^n b_n$ where $\{u_n\}$ is a sequence satisfying (2); then $\sum_0^\infty b_n$ is absolutely convergent. In virtue of I, the Cauchy sum $\sum_0^\infty c_n$ of the above series $\sum_0^\infty a_n$, $\sum_0^\infty b_n$ is not summable $[C, 0, \beta]$ due to [10].

Proof of Theorem 3. Since the case $\alpha + \beta = -1$ has been proved by Boyd [4] we may suppose that $\alpha + \beta > -1$. Let

$$f(x) = \frac{x^{a+1}}{\log\log x};$$

then, as $x \to \infty$,

$$f'(x) = \frac{(a+\beta+1)x^a}{\log\log x}(1 + O(1)) \tag{3.20}$$

and so there is a positive integer p such that

$$f(x+1) > f(x) > \text{ for } x \geq p$$

and let

$$\beta_n = \frac{(-1)^n}{\epsilon_n^a}(\delta_n - \delta_{p-1})\ (n \geq 0).$$

Then, for $n > p$,

$$\beta_n = \frac{(-1)^n}{\epsilon_n^a} f'(n-1+\theta)\ (0 < \theta < 1),$$

and so, by (3.20),

$$\beta_n = 0\left(\frac{1}{\log\,\log n}\right) = O(1)n \to \infty. \tag{3.21}$$

Now set

$$\alpha_0 = \alpha_1 = 0, \qquad \alpha_n = \frac{(-1)^n}{n\log n}\ (n \geq 2).$$

Then $\sum_0^\infty \alpha_n$ is summable $(C,-1,\beta)$, and $C_{\alpha,\beta}(B_n) = \beta_n \to 0$, $\sum_0^\infty b_n$ is summable $(C,\alpha,\beta)n$ to 0. Let

$$c_n = \sum_{r=0}^n a_r b_{n-r}, \quad Y_n = \sum_{r=0}^n c_r, \quad \sigma_n = C_\alpha(Y_n) \tag{3.22}$$

Then

$$Y_n = \sum_{r=0}^n a_{n-r}B_r = \sum_{r=0}^n a_{n-r,}\triangle^\alpha(\epsilon_r^\alpha \beta_r) = (-1)^n \sum_{r=0}^n a_{n-r,}|\delta_r - \delta_{r-1}| \tag{3.23}$$

and so, $n \to \infty$,

$$a(x) \to 0(c,r) \Rightarrow a(x) \to 0(c,r') \text{ for } r' > r \geq 0.$$

It follows that

$$b(x) \to 0(c,r) \Rightarrow b(x) \to 0(c,r') \text{ for } r' > r \geq 0 \tag{3.24}$$

and hence, by our Lemma, that

$$c(x) \to 0(c,\alpha+1,\beta) \Rightarrow c(x) \to 0(c,\alpha,\beta) \text{ for } \alpha > \beta \geq 0. \tag{3.25}$$

Necessary Condition: If $r = r' = -1$,the Theorem immediate follows from the summability of $(C,-1,\alpha,\beta)$. If $r > -1$, then by consistency Theorem for (C,r,α) summability (Gehring [3], Theorem 4.2.1]) it follows that both the functions $c(x)$ and $C_{\alpha,\beta}(x)$ are (C,α,β) convergent to s, see [11]. By [Hardy [5], Equation (6.1.6)], $S_r^n = S_{r+1}^n + \frac{1}{r+1}\frac{f(x)}{C_{\alpha,\beta}(x)}$, and the result follows since a linear combination of functions summable (C,k,α) to itself. The sufficient conditions to prove the theorem are: Consequently $\sum_0^\infty c_n$ is not summable $[C,\alpha+1,\beta]$ to 0. However, by a standard result ([7], Theorem 4), $\sum_0^\infty c_n$ is summable $[C,\alpha+1,\beta]$ to 0 and so, by the second inclusion in IV [2], the series cannot be summable $[C,\alpha+1,\beta]$ to any number other than 0. Hence $\sum_0^\infty c_n$ is not summable $[C,\alpha+1,\beta]$.

Remark. It is known in ([5], Theorem 6) that, given $\alpha \geq -1$, there are series $\sum_0^\infty \alpha_n$, $\sum_0^\infty b_n$, respectively summable $(C, -1, \beta)$ and (C, α, β) for which $\sum_0^\infty c_n$ is not summable (C, α, β).

Our Theorem 3 is stronger than this result, since (C, α, β) is included in, but is not equivalent to, $[C, \alpha + 1, \beta]$.

Institute of Engineering and Technology,
Lucknow-226028, India
E-mail: snmishra@ietlucknow.ac.in

References

[1] **Borwein, D.:** On multiplication of $(C, -u)$ summable series, Journal Londan Math. Soc., 33 (1958), 441 – 449.

[2] **Borwein, D. and Matsuoka, Y.:** On multiplication of Cesáro summable series, Quart. J. of Math. (Oxford). (2), 9 (1958), 310 – 316.

[3] **Polya, G.:** On strong and absolute summability, Proc. Glasgow Math. Assoc., 4 (1959), 122-139

[4] **Boyd, A.V.:** Multiplication of strongly summable series, Proc. Glasgow Math. Assoc., 7 (1960), 29-33.

[5] **Hardy, G. H.:** Divergent Series (Oxford, 1949).

[6] **Hyslop, J. M.:** Note on the strong summability of series, Proc. Glasgow Math. Assoc., 1 (1952-53), 16-20.

[7] **Winn, C. E.:** On the strong summability for any positive order. Math Zeitschrift, 37 (1933), 481-492.

[8] **Mishra, B. P. and Srivastava, A. P.:** Some remarks on absolute Summability of functions based on (C, α, β) Summability methods, Jour. Nat. Acad. Math., 25 (1982), 185-189.

[9] **Mishra, B. P. and Mishra, S. N.:** Strong Summability of functions based on $(D, k)(C, \alpha, \beta)$ Summability methods, Bull. Cal. Math. Soc., 99 (3)(2007), 305-322

[10] **Mishra, S. N. and Tripathi, Piyush:** A Gehring theorem for (C, α, β)convergence of Cesáro means of functions, International Journal-Mathematical Manuscripts, 9 No. 2 (2016), 203-206.

[11] **Mishra, S. N. and Tripathi, Piyush:** Cesáro mean and analysis of product summability methods, Global Journal of Pure & Applied Mathematics, 12 No. 1 (2016), 117-124.

[12] **Mishra, S. N.:** On double Fourier series and product summability methods, International Journal of Computational and Applied Mathematics, 12 No. 1 (2017), 125-128.