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Decomposition of Curvature Tensor Fields in Einstein-Sasakian First Order Recurrent Space

By

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Abstract

Takano [15] have studied decomposition of curvature tensor in a recurrent space. Sinha and Singh [14] have been studied and define decomposition of recurrent curvature tensor field in a finsler space. Negi and Rawat [2] and [12, 4] have study the decomposition of recurrent curvature tensor field in Kaehlerian space. Rawat and silswal[6] studied and define decomposition of the recurrent curvature tensor field in Tachibana space. Rawat and other author [5, 13], [10] and [1, 17] obtained various results on curvature tensor fields in various spaces in differential geometry. Various author[7] and [8, 9], obtained result on Käehlerian recurrent space. Further, Rawat and Chauhan [11] studied the decomposition of curvature tensor field in Einstein-Sasakian first order recurrent space of first order.

In present paper we have studied the decomposition of curvature tensor field R^h_{ijk} in terms of two non zero vectors and tensor field in Einstein-Sasakian first order recurrent space and several theorem have been established and proved.

Keywords: Sasakian space, Einstein space, Einstein-Sasakian space, recurrent space, Curvature tensor, Projective curvature tensor.

1. Introduction

An n-dimensional Sasakian space S_n (or normal contact metric space) is a Riemanian space which admits a unit killing vector fieldη_i satisfying (Okumura 1962) [3]

$$\Delta_i \Delta_j \eta_k = \eta_j g_{ik} - \eta_k g_{ij} \tag{1.1}$$

It is well known that the Sasakian space is Orientable and odd dimensional. Also we know that an n- dimensional Kaehlarian space $k - n$ is Riemann space which admits structure tensor field F^h_i satisfying

$$F^h_j F^i_h = -\delta^i_j \tag{1.2}$$

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$$F_{ij} = -F_{ji}, \qquad F_{ij} = F_i^a g_{aj} \tag{1.3}$$

and

$$F^h_{i,j} = 0 \tag{1.4}$$

where the comma (,) followed by an index denote the operator of covariant differentiation with respect to the metric tensor g_{ij} of Riemannian space.

The Riemannian curvature tensor field noted by R^h_{ijk} is given by

$$R^h_{ijk} = \partial_i\{^{\,h}_{jk}\} - \partial_j\{^{\,h}_{ik}\} + \{^{\,h}_{i\alpha}\}\{^{\,\alpha}_{jk}\} - \{^{\,h}_{j\alpha}\}\{^{\,\alpha}_{ik}\} \tag{1.5}$$

where $\delta_i = \frac{\delta}{\delta x^i}$.

Ricci tensor and the scalar curvature in S_n are respectively given by

$$R_{ij} = R^\alpha_{ij\alpha} \quad \text{and} \quad R = R_{ij} g^{ij} \tag{1.6}$$

If we define a tensor S_{ij} by

$$S_{ij} = F_i^\alpha R_{\alpha j} \tag{1.7}$$

then we have

$$S_{ij} = -S_{ji} \tag{1.8}$$

$$F_i^\alpha S_{\alpha j} = -S_{i\alpha} F_j^\alpha, \tag{1.9}$$

and

$$F_i^\alpha S_{jk,\alpha} = R_{ji,k} - R_{ki,j} \tag{1.10}$$

It has been verified in Yano [17] that the metric tensor g_{ij} and the Ricci tensor denoted by R_{ij} are hybrid in i and j.

Therefore, we get

$$g_{ij} = g_{rs} F_i^r F_j^t \tag{1.11}$$

and

$$R_{ij} = R_{rs} F_i^r F_j^t \tag{1.12}$$

The holomorphically projective curvature tensor P^h_{ijk} is define by

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{n+2}(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}S^h_i + 2S_{ij}F^h_k) \tag{1.13}$$

where $S_{ij} = F_i^\alpha R_{\alpha j}$.

Let us suppose that a Sasakian is Einstein one, and then the Ricci tensor satisfies $R_{ij} = \frac{R}{n} g_{ij}\, R_{,\alpha}$.

From which, we obtain

$$R_{ij,\alpha} = 0, \qquad S_{ij,\alpha} = 0 \qquad \text{and} \qquad S_{ij} = \frac{R}{n} F_{ij}.$$

The Bianchi identity for Einstein-Sasakian space are given

$$R^h_{ijk} + R^h_{jki} + R^h_{kij} = 0 \tag{1.14}$$

and

$$R^h_{ijk,\alpha} + R^h_{ik\alpha,j} + R^h_{i\alpha j,k} = 0 \tag{1.15}$$

$$T^i_{,jk} - T^i_{,kj} = T^\alpha R^i_{\alpha jk} = 0 \tag{1.16}$$

$$T^h_{i,ml} - T^h_{i,lm} = T^\alpha R^{it}_{\alpha ml} - T^h_\alpha R^\alpha_{iml} \tag{1.17}$$

An Einstein-Sasakian space is said to be Einstein-Sasakian first order recurrent space, if its curvature tensor field satisfy the condition

$$R^h_{ijk,\alpha} = \lambda_\alpha R^h_{ijk} \tag{1.18}$$

where λ_α is non zero vector and is known as recurrent vector field.

The following relations follow immediately from equation(1.14),

$$R_{ij,\alpha} = \lambda_\alpha R_{ij} \tag{1.19}$$

$$R_{,\alpha} = \lambda_\alpha R \tag{1.20}$$

2. Decomposition of recurrent curvature tensor R^h_{ijk}

Let us consider the decomposition of recurrent curvature tensor field R^h_{ijk} in the following way

$$R^h_{ijk} = \phi^h_\alpha A^\alpha \psi_{ijk} \tag{2.1}$$

where A^α, is non-zero vector field and ϕ^h_α, ψ_{ijk} are non-zero tensor fields, such that

$$\lambda_h \phi^h_a = Q_\alpha \tag{2.2}$$

and

$$\lambda_h A^h = 1 \tag{2.3}$$

Here, Q_α is called non-zero decomposed vector field.

Now, we shall prove the following:

Theorem 2.1. Using the decomposition (2.1), the Bianchi identities for R^h_{ijk} is expressed as in the following way

$$\psi_{ijk} + \psi_{jki} + \psi_{kij} = 0 \qquad (\psi_{ijk} = -\psi_{ikj}) \tag{2.4}$$

and

$$\lambda_\alpha \psi_{ijk} + \lambda_j \psi_{ik\alpha} + \lambda_k \psi_{i\alpha j} = 0. \tag{2.5}$$

Proof. By the equation (1.14), (1.15) and (2.1), we have

$$\phi^h_\alpha A^\alpha (\psi_{ijk} + \psi_{jki} + \psi_{kij}) = 0 \tag{2.6}$$

and

$$\phi^h_\alpha A^\alpha(\lambda_\alpha\psi_{ijk} + \lambda_j\psi_{ik\alpha} + \lambda_k\psi_{i\alpha j}) = 0 \tag{2.7}$$

The identities (2.4) and (2.5) follow immediately from these equation and the fact that $\phi^h_\alpha A^\alpha \neq 0$.

Theorem 2.2. The vector field a^α and the tensor field R^h_{ijk}, R_{ij}, ψ_{ijk} satisfies the following relation through the decomposition (2.1):
and

$$\lambda_\alpha R^\alpha_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik} = Q_\alpha A^\alpha \psi_{ijk} \tag{2.8}$$

Proof. By the help of Ricci identity, (1.18) and (1.19), we have

$$\lambda_\alpha R^\alpha_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik} \tag{2.9}$$

multiplying equation (2.1) by λ_h and from the equation (2.2), we obtain

$$\lambda_\alpha R^\alpha_{ijk} = Q_\alpha A^\alpha \psi_{ijk} \tag{2.10}$$

from equations (2.9) and (2.10), we get the required result (2.8).

Theorem 2.3. The quantities λ_α and field ϕ^h_α act as recurrent vector and recurrent tensor fields respectively and their recurrent relations can be written as in the following way through the decomposition (2.1):

$$\lambda_{\alpha,m} = \mu_m \lambda_\alpha \tag{2.11}$$

and

$$\phi^h_{\alpha,m} = \mu_m \phi^h_\alpha. \tag{2.12}$$

Proof. Firstly differentiating equation (2.8) covariantly with w.r.to x^m, then using (2.1) and (2.8), we obtain the relation

$$\lambda_{\alpha,m}\phi^h_\alpha A^\alpha \psi_{ijk} = \lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}. \tag{2.13}$$

Transvecting equation (2.13) with λ_α and using the equation (2.9), we get

$$\lambda_{\alpha,m}(\lambda_i R_{jk} - \lambda_j R_{ik}) = \lambda_\alpha(\lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}). \tag{2.14}$$

Again transvecting equation (2.13) with λ_h, we get

$$\lambda_{\alpha,m}(\lambda_i R_{jk} - \lambda_j R_{ik})\lambda_h = \lambda_\alpha \lambda_h(\lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}). \tag{2.15}$$

Since the right hand side of the equation (2.15) symmetric in α and h, then we have

$$\lambda_{\alpha,m}\lambda_h = \lambda_{h,m}\lambda_\alpha \tag{2.16}$$

provided that

$$\lambda_i R_{jk} - \lambda_j R_{ik} \neq 0.$$

The vector field $\lambda_\alpha \neq 0$, then there exist a proportional vector μ_m, such that $\lambda_{\alpha,m} = \mu_m \lambda_\alpha$.

Now differentiating (2.2) w.r.to x^m and from the condition (2.11), we find

$$\lambda_h \phi^h_{\alpha,m} = (Q_{\alpha m} - \mu_m Q_\alpha \tag{2.17}$$

From above equation, it is clear that

$$\lambda_h \phi^h_{\alpha,m} = \lambda_\alpha \phi^\alpha_{\alpha,m} \tag{2.18}$$

Since ,$\lambda_\alpha \neq 0$ is recurrent vector field, then we can get a proportional vector field μ_m such that $\phi^h_{\alpha,m} = \mu_m \phi^h_\alpha$
which complete the proof.

Theorem 2.4. The decomposition vector field Q_α and the tensor field ϕ^h_α, act as recurrent vector field and recurrent tensor fields respectively and their recurrent form can be written as in the following way through the decomposition (2.1):

$$Q_{\alpha,m} = 2\mu_m Q_\alpha \tag{2.19}$$

and

$$(\lambda_m - 2\mu_m)\psi_{ijk} = \psi_{i,k,m} \tag{2.20}$$

Proof. Differentiating (2.2) covariantly w.r.to x^m, and using the equation (2.2), (2.11) and (2.12), we obtain the required result (2.19). Further, differentiating (2.1) covariantly w.r.to x^m, and using (1.18), (2.2), (2.11) and (2.12), we obtain the recurrent form (2.20).

Theorem 2.5. The curvature tensor R^h_{ijk} is equal to holomorphically projective curvature tensor field under the decomposition (2.1), if

$$\delta^h_j \psi_{ik} - \delta^h_i \psi_{jk} + \psi_{\alpha k}(F^h_j F^\alpha_i - F^h_i F^\alpha_j) + 2F^h_k F^\alpha_i \psi_{\alpha j} = 0. \tag{2.21}$$

Proof. Contract the indices h and k in the relation (2.1), we have equation (1.13) may be written as

$$P^h_{ijk} = R^h_{ijk} + D^h_{ijk} \tag{2.22}$$

where

$$D^h_{ijk} = \frac{1}{n+2}(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}S^h_i + 2S_{ij}F^h_k) \tag{2.23}$$

$$R_{ij} = \phi^k_\alpha A^\alpha \psi_{ijk}. \tag{2.24}$$

In view of equation (2.22), we have

$$S_{ij} = F^\alpha_i R_{\alpha j} = F^\alpha_i \phi^r_\alpha A^\alpha \psi_{\alpha jr} \tag{2.25}$$

using the relations equations (2.22) and (2.23) in (2.23), we find

$$D^h_{ijk} = \frac{\phi^r_\alpha A^\alpha}{n+2}\{\psi_{ikr}\delta^h_j - \psi_{jkr}\delta^h_i + \psi_{\alpha kr}(F^h_j F^\alpha_i - F^h_i F^\alpha_j) + 2F^h_k F^\alpha_i \psi_{\alpha jr}\}. \quad (2.26)$$

If in (2.22) $D^h_{ijk} = 0$, then $P^h_{ijk} = R^h_{ijk}$, so equate the equation (2.24) to zero, we have

$$\psi_{ikr}\delta^h_j - \psi_{jkr}\delta^h_i + \psi_{\alpha kr}(F^h_j F^\alpha_i - F^h_i F^\alpha_j) + 2F^h_k F^\alpha_i \psi_{\alpha jr} = 0 \quad (2.27)$$

multiplying the above equation by A^α and using the relation $\phi_{ijk}A^k = \phi_{ij}$, we get the required relation.

Theorem 2.6. Using the decomposition (2.1), the scalar curvature R satisfies the following relation:

$$\lambda_k R = g^{ij} Q_\alpha A^\alpha \psi_{ijk} \quad (2.28)$$

or

$$R_{,k} = g^{ij} Q_\alpha A^\alpha \psi_{ijk}.$$

Proof. Multiplying (2.22) by g^{ij} on both sides, we obtain

$$g^{ij} R_{ij} = g^{ij} \phi^k_\alpha A^\alpha \psi_{ijk} \quad (2.29)$$

$R = g^{ij} \phi^k_\alpha A^\alpha \psi_{ijk}$.

Now transvecting (2.28) with λ_k and using (2.2), we get

$$\lambda_k R = g^{ij} Q_\alpha A^\alpha \psi_{ijk}.$$

which complete the proof of the theorem.

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