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Bayesian Estimation of Power Function Distribution using Lower Record Values

By

Rajiv Saksena, Vivek Tyagi[1], Ritu Kumari[2] and Ashok Kumar*

Abstract

This paper considering the Power function distribution as a life time model. The Bayesian estimation for the parameter, reliability function and hazard rate of power function distribution is obtained in case of lower record values. This study provides the Bayesian estimation under both informative (gamma) and non-informative (quasi, uniform, Jeffreys') priors. Both symmetric (squared error loss function) and asymmetric (generalized entropy loss function) loss functions are also considered for the Bayesian estimation.

Keywords : Power function distribution, Lower record values, Priors, Loss functions.
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1. Introduction

Meniconi and Barry [10] showed that power function distribution is useful to assess electrical component reliability. Power function distribution shows a better fit for failure time data and offers more suitable information about reliability and hazard rates. Due to this property, power function distribution is preferred over lognormal, exponential and Weibull distributions. Dallas [4] has been showed that if X follows power distribution, then $1/X$ follows the Pareto distribution. Rahman et al. [12] have used different symmetric and asymmetric loss functions to get the Bayes estimators for the parameter of power function distribution using complete sample. Kifayat et al. [8] compared the Bayes estimators of the parameter of power function distribution for the informative and non-informative priors.

This paper considers the Bayesian estimation of the parameter, reliability function and hazard rate of the power function distribution. For the Bayesian inference, both informative and non-informative priors are considered. The sym-

*Corresponding author
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metric and asymmetric loss functions are also taken into consideration.

The scheme of lower record values is adopted for the data collection, the thrust of the study is the Bayesian estimation of the power function distribution based on lower record values. The concept of record values has been discussed widely in the literature for Bayesian estimation by several authors viz. Soliman et al. [14]; Tarvirdizade [15]; Nadar and Kizilaslan [11]; Singh et al. [13]. Baklizi, A. [2] have discussed the likelihood and Bayesian estimation of stress-strength reliability of generalized exponential distribution using lower record value. Hassan et al. [7] considered the estimation of stress- strength reliability using lower record values from exponentiated inverted Weibull distribution. Kumari et al. [9] have discussed the Bayesian estimation of stress-strength reliability for generalized inverted exponential distribution using upper record values.

Record values played a measure role in daily life problems concerning data relating to numerous fields such as economics, weather, sports etc. Chandler [3] introduced the main idea of record values, inter-record times and started the statistical study of record values as a model for successive extremes in a sequence of independently and identically distributed random variables. The record values can be classified into two categories, lower record values and upper record values. An observation X_j will be called an upper record value if its value is greater than all of previous observations i.e.,$X_j > X_i$ for every $j > i$ and it will be called a lower record value if its value is less than all of previous observations i.e.,$X_j < X_i$ for every $j > i$.

A continuous random variable X is said to have a Power function distribution, if its probability density function (pdf) is

$$f(x) = \theta x^{\theta-1}; \qquad 0 < x < 1, \theta > 0,$$

where θ is shape parameter.

The corresponding cumulative density function (cdf) is

$$F(x) = x^{\theta}; \qquad 0 < x < 1,\ \theta > 0.$$

The reliability function of the power function distribution is given by

$$R(x) = 1 - x^{\theta}.$$

The hazard rate of the power function distribution is given by

$$h(x) = \frac{\theta x^{\theta-1}}{1 - x^{\theta}}.$$

2. Likelihood Function

Let $(x_{(1)}, x_{(2)}, ..., x_{(m)})$ be the m lower record values i.e., $x_{(1)} \geq x_{(2)} \geq \ldots \geq x_{(m)}$ from a population with pdf $f(x)$ and cdf $F(x)$, then, the likelihood function is defined as (Arnold et al. [1])

$$L = f(x_{(m)}) \prod_{i=1}^{m-1} \frac{f(x_{(i)})}{F(x_{(i)})}.$$

Let $(x_{(1)}, x_{(2)}, ..., x_{(m)})$ be the m lower record values from power function distribution, the likelihood function is

$$L(\underset{\sim}{x}; \theta) = \frac{\theta^m x_{(m)}^{\theta}}{\prod_{i=1}^{m} x_{(i)}}.$$

3. Bayesian Estimation

In this Section, Bayes estimator are derived using informative and non-informative priors under both symmetric and asymmetric loss functions. A brief introduction about loss function is given below:

Squared Error Loss Function

The squared error loss function (SELF) is the simplest symmetric loss function and it is defined as $L(\hat{\theta}, \theta) \propto (\theta - \hat{\theta})^2$ where $\hat{\theta}$ is the Bayes estimator of unknown parameter θ. The Bayes estimator of θ under SELF is $\hat{\theta} = E(\theta | \underset{\sim}{x})$, where expectation is taken with respect to posterior density.

General Entropy Loss Function

Generalized entropy loss function (GELF) is an asymmetric loss function and it is suggested by Calabria and Pulcini (1996). This loss function is a generalization of the entropy loss function and defined as

$$L(\theta, \hat{\theta}) \propto \left(\frac{\hat{\theta}}{\theta}\right)^c - c \ln\left(\frac{\hat{\theta}}{\theta}\right) - 1,$$

where $c \neq 0$. If $c < 0$, then, under estimation of the parameter gets more serious than over estimation and vice-versa. Bayes estimator of θ under GELF is

$$\hat{\theta} = [E(\theta^{-c} | \underset{\sim}{x})]^{(-1/c)}.$$

For prior information, we have considered two types of priors, one is informative prior and the other is non-informative prior. A brief introduction of prior distribution is given below:

Non-Informative Prior

Non-informative priors are used when no prior information or a little information is available about the parameter. Here, we have considered, three types of non-informative priors viz. uniform prior, quasi prior, Jeffreys' prior as discussed below:

(i) **Uniform Prior:** Uniform prior for the parameter θ is given by

$$g_1(\theta) \propto 1.$$

(ii) **Quasi Prior:** Quasi prior for the parameter θ is given by

$$g_2(\theta) \propto \frac{1}{\theta^d}; \qquad d \geq 0, \theta > 0.$$

(iii) **Jeffreys' Prior:** Jeffreys' (1961) provides a method to select a non-informative prior for the parameter θ and obtained prior from this method is called Jefferys' prior. Jefferys' prior is based on Fisher information criteria of the model. Jefferys' prior for single parameter θ is defined as

$$g(\theta) \propto \sqrt{I(\theta)},$$

where $I(\theta) = -E_\theta \left\{ \frac{\partial^2 \ln L(\underline{x};\theta)}{\partial \theta^2} \right\}$, is the Fisher information based on likelihood function $L(\underline{x};\theta)$.

In case of power function distribution, Jeffreys' prior is

$$g_3(\theta) \propto \frac{1}{\theta}; \quad \theta > 0.$$

Informative Prior

The most common used informative prior is the gamma prior and is given by

$$g_4(\theta) \propto \theta^{a_1-1} e^{-b_1\theta}; \quad \theta > 0, a_1, b_1 > 0.$$

where a_1 and b_1 are the hyper-parameters.

Credible Intervals

In Bayesian statistics, a credible interval is an interval in the domain of a posterior probability distribution. The $100(1-\alpha)\%$ equal tail credible interval for exact posterior distribution can be defined as (Eberly and Casella [6])

$$P(\theta < L) = \int_{-\infty}^{L} \pi(\theta|\underset{\sim}{x})\, d\theta = \frac{\alpha}{2}, \quad P(\theta > U) = \int_{U}^{\infty} \pi(\theta|\underset{\sim}{x})\, d\theta = \frac{\alpha}{2}$$

where $\pi(\theta|\underset{\sim}{x})$ is the posterior density of θ and (L, U) are the lower limit and upper limit of the credible interval, respectively.

3.1. Bayesian Estimation of θ, R(t) and h(t) using gamma prior under SELF and GELF

The posterior distribution of the unknown parameter θ in case of gamma prior is

$$\pi(\theta|\underset{\sim}{x}) = \frac{L(\underset{\sim}{x};\theta)g_4(\theta)}{\int_0^\infty L(\underset{\sim}{x};\theta)g_4(\theta)d\theta}$$

$$= \frac{\theta^{m+a_1-1}e^{-b_1\theta}x_{(m)}^{\theta}\left\{b_1+\ln\left(\frac{1}{x_{(m)}}\right)\right\}^{m+a_1}}{\Gamma(m+a_1)}.$$

Bayes estimator of θ in case of gamma prior under SELF is given by

$$\hat{\theta}_{GS} = \int_0^\infty \theta\pi(\theta|\underset{\sim}{x})\,d\theta = \frac{m+a_1}{b_1+\ln\left(\frac{1}{x_{(m)}}\right)}.$$

Bayes estimator of $R(t)$ under SELF is given by

$$\hat{R}(t)_{GS} = \int_0^\infty R(t)\pi(\theta|\underset{\sim}{x})d\theta = 1-\left\{\frac{b_1+\ln\left(\frac{1}{x_{(m)}}\right)}{b_1+\ln\left(\frac{1}{tx_{(m)}}\right)}\right\}^{m+a_1}.$$

Bayes estimator of $h(t)$ under SELF is given by

$$\hat{h}(t)_{GS} = \int_0^\infty h(t)\pi(\theta|\underset{\sim}{x})d\theta$$

$$= \frac{(m+a_1)\left\{b_1+\ln\left(\frac{1}{x_{(m)}}\right)\right\}^{m+a_1}}{t}\sum_{j=0}^{\infty}\frac{1}{\left\{b_1+\ln\left(\frac{1}{t^{j+1}x_{(m)}}\right)\right\}^{m+a_1+1}}.$$

Bayes estimator of θ under GELF is given by

$$\hat{\theta}_{GG} = \left[\int_0^\infty \theta^{-c}\pi(\theta|\underset{\sim}{x})d\theta\right]^{(-1/c)} = \frac{1}{b_1+\ln\left(\frac{1}{x_{(m)}}\right)}\left[\frac{\Gamma(m+a_1-c)}{\Gamma(m+a_1)}\right]^{(-1/c)}.$$

Bayes estimator of $R(t)$ under GELF is given by

$$\hat{R}(t)_{GG} = \left[\int_0^\infty (1-t^{\theta})^{-c}\pi(\theta|\underset{\sim}{x})d\theta\right]^{(-1/c)}$$

$$= \left[\left\{b_1+\ln\left(\frac{1}{x_{(m)}}\right)\right\}^{(m+a_1)}\sum_{j=0}^{\infty}\frac{\begin{pmatrix}-c\\ j\end{pmatrix}(-1)^j}{\left\{b_1+\ln\left(\frac{1}{t^jx_{(m)}}\right)\right\}^{(m+a_1)}}\right]^{(-1/c)}.$$

Bayes estimator of $h(t)$ under GELF is given by

$$\hat{h}(t)_{GG} = \left[\int_0^\infty \left(\frac{\theta t^{\theta-1}}{1-t^\theta}\right)^{-c} \pi(\theta|\underset{\sim}{x})d\theta\right]^{-1/c}$$

$$= \left[\frac{\left\{b_1 + \ln\left(\frac{1}{x_{(m)}}\right)\right\}^{(m+a_1)}}{\Gamma(m+a_1)} \sum_{j=0}^{\infty} \frac{\binom{c}{j}(-1)^j t^c \Gamma(m+a_1-c)}{\left\{b_1 + \ln\left(\frac{t^{c-j}}{x_{(m)}}\right)\right\}^{(m+a_1-c)}}\right]^{(-1/c)}.$$

3.2. Bayesian estimation in case of Quasi Prior, Jeffreys' Prior and Uniform Prior under SELF and GELF

Quasi, Jeffreys' and Uniform priors can be obtained from gamma prior by putting the particular values of the hyper-parameters as given below:

(i) If $a_1 = 1 - d$ and $b_1 = 0$, gamma prior provides Quasi prior.
(ii) If $a_1 = 0$ and $b_1 = 0$, gamma prior provides Jeffreys' prior.
(iii) If $a_1 = 1$ and $b_1 = 0$, gamma prior provides Uniform prior.

Bayes estimators of θ, $R(t)$ and $h(t)$ using Quasi prior under SELF are given by

$$\hat{\theta}_{QS} = \frac{m-d+1}{\ln\left(\frac{1}{x_{(m)}}\right)},$$

$$\hat{R}(t)_{QS} = 1 - \left\{\frac{\ln\left(\frac{1}{x_{(m)}}\right)}{\ln\left(\frac{1}{tx_{(m)}}\right)}\right\}^{m-d+1},$$

and

$$\hat{h}(t)_{QS} = \frac{(m-d+1)\left[\ln\left(\frac{1}{x_{(m)}}\right)\right]^{(m-d+1)}}{t} \sum_{j=0}^{\infty} \frac{1}{\left[\ln\left(\frac{1}{x_{(m)}t^{j+1}}\right)\right]^{(m-d+2)}}.$$

Bayes estimators of θ, $R(t)$ and $h(t)$ using Quasi prior under GELF are given by

$$\hat{\theta}_{QG} = \frac{1}{\ln\left(\frac{1}{x_{(m)}}\right)}\left[\frac{\Gamma(m-d-c+1)}{\Gamma(m-d+1)}\right]^{(-1/c)},$$

$$\hat{R}(t)_{QG} = \left[\left\{\ln\left(\frac{1}{x_{(m)}}\right)\right\}^{m-d+1} \sum_{j=0}^{\infty} \frac{\binom{-c}{j}(-1)^j}{\left\{\ln\left(\frac{1}{t^j x_{(m)}}\right)\right\}^{m-d+1}}\right]^{(-1/c)},$$

and

$$\hat{h}(t)_{QG} = \left[\frac{\left\{\ln\left(\frac{1}{x_{(m)}}\right)\right\}^{m-d+1}}{\Gamma(m-d+1)} \sum_{j=0}^{\infty} \frac{\begin{pmatrix} c \\ j \end{pmatrix} (-1)^j t^c \Gamma(m-d-c+1)}{\left\{\ln\left(\frac{t^{c-j}}{x_{(m)}}\right)\right\}^{m-d-c+1}} \right]^{(-1/c)}.$$

Bayes estimators of θ, $R(t)$ and $h(t)$ using Jeffreys' prior under SELF are given by

$$\hat{\theta}_{JS} = \frac{m}{\ln\left(\frac{1}{x_{(m)}}\right)},$$

$$\hat{R}(t)_{JS} = 1 - \left\{ \frac{\ln\left(\frac{1}{x_{(m)}}\right)}{\ln\left(\frac{1}{tx_{(m)}}\right)} \right\}^m,$$

$$\hat{h}(t)_{JS} = \frac{m\left\{\ln\left(\frac{1}{x_{(m)}}\right)\right\}^m}{t} \sum_{j=0}^{\infty} \frac{1}{\left\{\ln\left(\frac{1}{t^{j+1}x_{(m)}}\right)\right\}^{m+1}}.$$

Bayes estimators of θ, $R(t)$ and $h(t)$ using Jeffreys' prior under GELF are given by

$$\hat{\theta}_{JG} = \frac{1}{\ln\left(\frac{1}{x_{(m)}}\right)} \left[\frac{\Gamma(m-c)}{\Gamma(m)}\right]^{(-1/c)},$$

$$\hat{R}(t)_{JG} = \left[\left\{\ln\left(\frac{1}{x_{(m)}}\right)\right\}^{(m)} \sum_{j=0}^{\infty} \frac{\begin{pmatrix} -c \\ j \end{pmatrix} (-1)^j}{\left\{\ln\left(\frac{1}{t^j x_{(m)}}\right)\right\}^{(m)}} \right]^{(-1/c)},$$

$$\hat{h}(t)_{JG} = \left[\frac{\left\{\ln\left(\frac{1}{x_{(m)}}\right)\right\}^m}{\Gamma(m)} \sum_{j=0}^{\infty} \frac{\begin{pmatrix} c \\ j \end{pmatrix} (-1)^j t^c \Gamma(m-c)}{\left\{\ln\left(\frac{t^{c-j}}{x_{(m)}}\right)\right\}^{(m-c)}} \right]^{(-1/c)}.$$

Bayes estimators of θ, $R(t)$ and $h(t)$ using Uniform prior under SELF are given by

$$\hat{\theta}_{US} = \frac{m+1}{\ln\left(\frac{1}{x_{(m)}}\right)},$$

$$\hat{R}(t)_{US} = 1 - \left\{ \frac{\ln\left(\frac{1}{x_{(m)}}\right)}{\ln\left(\frac{1}{tx_{(m)}}\right)} \right\}^{m+1},$$

$$\hat{h}(t)_{US} = \frac{(m+1)\left\{\ln\left(\frac{1}{x_{(m)}}\right)\right\}^{m+1}}{t}\sum_{j=0}^{\infty}\frac{1}{\left\{\ln\left(\frac{1}{t^{j+1}x_{(m)}}\right)\right\}^{m+2}}.$$

Bayes estimators of θ, $R(t)$ and $h(t)$ using Uniform prior under GELF are given by

$$\hat{\theta}_{UG} = \frac{1}{\ln\left(\frac{1}{x_{(m)}}\right)}\left[\frac{\Gamma(m+1-c)}{\Gamma(m+1)}\right]^{(-1/c)}.$$

$$\hat{R}(t)_{UG} = \left[\left\{\ln\left(\frac{1}{x_{(m)}}\right)\right\}^{(m+1)}\sum_{j=0}^{\infty}\frac{\begin{pmatrix}-c\\ j\end{pmatrix}(-1)^j}{\left\{\ln\left(\frac{1}{t^j x_{(m)}}\right)\right\}^{(m+1)}}\right]^{(-1/c)},$$

$$\hat{h}(t)_{UG} = \left[\frac{\left\{\ln\left(\frac{1}{x_{(m)}}\right)\right\}^{(m+1)}}{\Gamma(m+1)}\sum_{j=0}^{\infty}\frac{\begin{pmatrix}c\\ j\end{pmatrix}(-1)^j t^c\Gamma(m+1-c)}{\left\{\ln\left(\frac{t^{c-j}}{x_{(m)}}\right)\right\}^{(m+1-c)}}\right]^{(-1/c)}.$$

4. Simulation

In this section, a simulation study is conducted to compare the performance of proposed estimators in terms of risk function and length of credible intervals. Simulation study is conducted based on the 3000 replications. Three different sample sizes are considered viz., small ($n = 20$), medium ($n = 30$), and large ($n = 50$). In case of informative prior, the hyper-parameters are chosen as prior mean is equal to true value of parameter. The risk function and length of credible intervals of Bayes estimates using informative prior of θ, $R(t)$ and $h(t)$ are presented in Table 1. Table 2 listed the risk function and length of credible intervals of Bayes estimates using non-informative prior of θ, $R(t)$ and $h(t)$.

Table 1: Risk function and length of credible intervals of Bayes estimates using informative prior for $\theta = 2$, $t = 0.5$.

	n	SELF	GELF $c = 2.0$	GELF $c = -2.0$	Length of credible intervals
θ	20	0.0437	0.0266	0.0262	0.1563
	30	0.0364	0.0183	0.0196	0.1514
	50	0.0313	0.0123	0.0134	0.1267
$R(t)$	20	0.0597	0.0361	0.0352	0.1559
	30	0.0476	0.0282	0.0194	0.1512
	50	0.0367	0.0171	0.0132	0.1309
$h(t)$	20	0.0410	0.0222	0.0267	0.1873
	30	0.0360	0.0192	0.0205	0.1814
	50	0.0291	0.0161	0.0252	0.1636

Table 2: Risk function and length of credible intervals of Bayes estimates using non-informative prior for $\theta = 2$, $t = 0.5$.

	n	SELF	GELF $c = 2.0$	GELF $c = -2.0$	Length of credible intervals
θ	20	0.1914	0.1670	0.1720	0.2571
	30	0.1781	0.1614	0.1453	0.2120
	50	0.1568	0.1105	0.1323	0.1842
R(t)	20	0.1945	0.1695	0.1649	0.2194
	30	0.1831	0.1624	0.1602	0.1911
	50	0.1698	0.1473	0.1432	0.1683
h(t)	20	0.2044	0.1823	0.1928	0.2230
	30	0.1940	0.1677	0.1715	0.2012
	50	0.1541	0.1400	0.1449	0.1743

It can be seen from Tables 1-2, the length of credible intervals decreases as the sample size increases. It can also be observed that the risk function and length of credible intervals in case of informative priors smaller than the non-informative priors.

5. Conclusion

This paper considered the Power function distribution as a life time model. The Bayesian estimation for the parameter, reliability function and hazard rate of power function distribution is done in case of lower record values. This study provides the Bayesian estimation under both informative (gamma) and non-informative (quasi, uniform, Jeffreys') priors. Both symmetric (squared error loss function) and asymmetric (generalized entropy loss function) loss functions is considered for the Bayesian estimation. Bayes estimators are obtained in closed form under both informative (gamma) and non-informative (quasi, uniform, Jeffreys') priors. A simulation study is also conducted to compare the performance of proposed estimators.

Department of Statistics
Lucknow University, Lucknow
∗e-mail: stats.ashok@gmail.com
[1]Department of Statistics,
NAS College, Meerut
[2]Department of Statistics,
Panjab University, Chandigarh

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