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Generalized CR-Warped Product Submanifolds of Nearly LP-Sasakian Manifolds with a Connection

By

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Abstract
The aim of this paper is to study generalized CR-warped productof the type $M = N_\perp \times_f N_T$and $M = N_T \times_f N_\perp$, where N_T and $N_\perp$ are invariant and anti-invariant submanifolds and general sharp inequality, namely $\|h\|^2 \geq \frac{2}{9}s + \frac{2}{9}s\,\|\nabla lnf\|^2$, for contact CR-warped products submanifoldsof nearly Lorentzian para-Sasakian manifolds with semi symmetric semimetric connection.

Keywords: Contact CR-submanifolds, warped product, Contact CR-warped product,Nearly Lorentzian para-Sasakian structure, semi symmetric semi metric connection.
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1. Introduction

A Lorentzian para-Sasakian manifold was studied by Matsumoto, K [12]. Then Mihai, I et al. [14] introduced the same notion independently and they obtained several results on this manifolds. Lorentzian para-Sasakian manifolds have also been studied by De, U.C. et al., [9] and others. Lorentzian para-Sasakian manifold with different connections were studied by several authors in ([1, 2, 15, 16]). In [19], Rahman S and et al., studied semi-invariant submanifolds of a nearly Lorentzian para-Sasakian manifold.

The concept and definition of warped product manifolds was given by Bishop and O'Neill [5].These manifolds arethe generalisationof Riemannian product manifolds and appear in differential geometric studies in natural way [7, 10]. Later on, CR-warped product submanifolds of Kaehler manifolds were studied by Chen, B.Y [8] and showed many interesting results on the existence of warped product and proved general sharp inequalities for the second fundamental form in terms of the warping function f. Arslan, K and et al. [4], Bonanzinga, V., Matsumoto, K [6] and Mihai, I. [13] studied for the same inequalities in almost Hermitian as well as almost contact metric manifolds. Also B. Sahin studied Non-existence of warped product semi slant submanifold of Kaehler manifolds [20].Contact CR-warped product submanifolds of nearly Lorentzian para-Sasakian manifold were studied by Rahman S [17].

Our aim in this paper is to study the warped product contact CR-submanifolds of nearly Lorentzian para-Sasakian manifolds with semi symmetric semi metric connection. This paper is organized as; section 2 is devoted to preliminaries. In section 3,we proved some existence and nonexistence results. Section 4 deals a general sharp inequality for the second fundamental form in terms of the warping function f on nearly Lorentzian para-Sasakian manifolds with semi symmetric semi metric connection. The inequali-

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ty obtained in this section is more general as it generalizes all inequalities obtained for contact CR-warped products in contact metric manifolds.

2. Preliminaries

Let $\bar{M}$ be an n-dimensional almost contact metric manifold with almost contact metric structure (ϕ, ξ, η, g) such that

$$\phi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1, g(X, \xi) = \eta(X) \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

$$g(\phi X, Y) = g(X, \phi Y) = \psi(X, Y) \tag{2.2}$$

For vector fields X, Y tangent to M.Then the structure$(\phi, \xi, \eta, g)$is term as Lorentzianpara-contact structure.Also in a Lorentzian para-contact structure the following relations hold:

$\phi\xi = 0, \quad \eta(\phi X) = 0,$ and $rank(\phi) = n - 1$

A Lorentzian para-contact manifold $\bar{M}$ is called Lorentzian para-Sasakian (LP-Sasakian) manifold if

$$(\bar{\nabla}_X \phi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi \tag{2.3}$$

$$\bar{\nabla}_X \xi = \phi X \tag{2.4}$$

For all vector fields X, Y tangent to $\bar{M}$ where $\bar{\nabla}$ is the Riemannian connection with respect to g. Further, an almost contact metric manifold $\bar{M}$ on (ϕ, ξ, η, g) is called nearly Lorentzian para-Sasakian if

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = 2g(X, Y)\xi + 4\eta(X)\eta(Y) + \eta(X)Y + \eta(Y)X \tag{2.5}$$

The covariant derivative of the tensor field $\phi$is defined as

$$(\bar{\bar{\nabla}}_X \phi)Y = \bar{\bar{\nabla}}_X \phi Y - \phi \bar{\bar{\nabla}}_X Y \tag{2.6}$$

On the other hand, semi symmetric semi metric connection$\bar{\bar{\nabla}}$ defined by

$$\bar{\bar{\nabla}}_X Y = \nabla_X Y - \eta(Y)X + g(X, Y)\xi \tag{2.7}$$

$$(\bar{\bar{\nabla}}_X g)(Y, Z) = -\eta(Y)g(X, Z) - \eta(Z)g(X, Y) \tag{2.8}$$

Substituting (2.1), (2.2) and (2.4) in (2.6) and (2.7) respectively, we get

$$(\bar{\bar{\nabla}}_X \phi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y) - 3\eta(X)\phi Y + g(X, \phi Y)\xi \tag{2.9}$$

In particular, an almost contact metric manifold $\bar{M}$ on (ϕ, ξ, η, g) is called nearly Lorentzian para-Sasakian manifold $\bar{M}$with semi symmetricsemi metric connection if

$$(\bar{\bar{\nabla}}_X \phi)Y + (\bar{\bar{\nabla}}_Y \phi)X = 2g(X, Y)\xi + \eta(Y)X + \eta(X)Y + 4\eta(X)\eta(Y)\xi - 3\eta(X)\phi Y - 3\eta(Y)\phi X + 2g(X, \phi Y)\xi. \tag{2.10}$$

Now, let M be a submanifold immersed in $\bar{M}$. The Riemannian metric induced on M is denoted by the same symbol g. Let TM and $T^{\perp}M$ be the Lie algebras of vector fields tangential to M and normal to M respectively and ∇ be the induced Levi-Civita connection on M, then the Gauss and Weingarten formulas are given by

$$\bar{\bar{\nabla}}_X Y = \nabla_X Y + h(X, Y) \tag{2.11}$$

$$\bar{\bar{\nabla}}_X N = -A_N X + \nabla_X^{\perp} N - \eta(X)N \tag{2.12}$$

For any $X, Y \in TM$ and $N \in T^{\perp}M$where $\nabla^{\perp}$ is the connection on the normal bundle $T^{\perp}M$, h is the second fundamental form and A_N is the Weingarten map associated with N as

$g(A_N X, Y) = g(h(X, Y), N)$ (2.13)

In 1969 Bishop and O ̉Niell [5] introduced the notion of warped product manifolds. They defined as:
Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and f be a positive differentiable function on N_1. The warped product of N_1 and N_2 is the Riemannian manifold
$N_1 \times_f N_2 = (N_1 \times N_2, g)$, where $g = g_1 + f^2 g_2$

A warped product manifold $N_1 \times_f N_2$ is said to be trivial if the warping function f is constant. We recall the following general result for later use.

Lemma 2.1: ([5]). Let $M = N_1 \times_f N_2$ be a warped product manifold with the warping function, then
(i) $\nabla_X Y \in \Gamma(TN_1)$ is the lift of $\nabla_X Y$ on N_1,
(ii) $\nabla_X Z = \nabla_Z X = (X \ln f)Z$
(iii) $\nabla_Z \omega = \nabla_Z^{N_2} \omega - g(Z, \omega)\nabla \ln f$

for each $X, Y \in \Gamma(TN_1)$ and $Z, \omega \in \Gamma(TN_2)$,where $\nabla \ln f$is the gradient of lnf and ∇ and ∇^{N_2} denote the Levi Civita connections on M and N_2, respectively.

For a Riemannian manifold M of dimension n and a smooth function f on M, we recall ∇f, the gradient of f which is defined by

$g(\nabla f, X) = X(f)$ ….. (2.14)

for any $X \in \Gamma(TM)$. As a consequence, we have

$\|\nabla f\|^2 = \sum_{i=1}^{n}(e_i(f))^2$ ….. (2.15)

for an orthonormal frame $\{e_1, \dots, e_n\}$ on M.

3. Contact CR-warped product Submanifolds

In this section first we recall the invariant, anti-invariant and contact CR-submanifolds. For submanifolds tangent to the structure vectorfield ξ,there are different classes of submanifolds. We mention the following:
(i) A submanifold M tangent to ξ is an invariant submanifold if ϕ preserves any tangent space of M, that is $\phi(T_pM) \subset T_pM$, foreverY$p \in M$.
(ii) A submanifold M tangent to ξ is an anti-invariant submanifold if ϕ maps any tangent space of M into the normal space, that is, $\phi(T_pM) \subset T_p^{\perp}M$. For every $p \in M$.

Let M be a Riemannian manifold isometrically immersed in an almost contact metric manifold $\bar{M}$, then for every $p \in M$ there exists a maximal invariant subspace denoted by $D_p$of the tangent space T_pM of M. If the dimension of D_p is same for all values of $p \in M$, then D_p gives an invariant distribution D on M.

A submanifold M of an almost contact manifold $\bar{M}$ is said to be a contact CR submanifold if there exists on M a differentiable distribution D whose orthogonal complementary distribution $D^{\perp}$ is anti-invariant, that is;
(i) $TM = D \oplus D^{\perp} \oplus \langle \xi \rangle$
(ii) D is invariant distribution,i.e. $\phi D \subseteq TM$,

(iii) $D^{\perp}$ is an anti-invariant distribution i.e.$\phi D^{\perp} \subseteq T^{\perp}M$.

A contact CR-submanifold is an anti-invariant if $D_p = \{0\}$ and invariant if $D_p^{\perp} = \{0\}$ respectively, for every $p \in M$. It is a proper contact CR-submanifold if neither $D_p = \{0\}$ nor $D_p^{\perp} = \{0\}$, for ry $p \in M$.

If v is the ϕ-invariant subspace of the normal bundle $T^{\perp}M$, then in case of contact CR- submanifold , the normal bundle $T^{\perp}M$ can be decomposed as

$$T^{\perp}M = \phi D^{\perp} \oplus v$$

Where v and ϕ-invariant normal sub bundle of $T^{\perp}M$.

In this section, we investigate the warped products $M = N_{\perp} \times_f N_T$ and $M = N_T \times_f N_{\perp}$ where N_T and $N_{\perp}$ are invariant and anti-invariant Submanifolds of a nearly Lorentzian para-Sasakian fold$\bar{M}$,respectively.First we discuss the warped products $M = N_{\perp} \times_f N_T$, here two possible cases arise:

(i) ξ is tangent to N_T,

(ii) ξ is tangent to $N_{\perp}$,

We start with the case (i)

Theorem 3.1: Let $\bar{M}$ be a nearly Lorentzian para-Sasakian manifold with a semi symmetric semimetric connection. Then there do not exist warped product submanifold $M = N_{\perp} \times_f N_T$such that N_T is an invariant submanifold tangent to ξ and $N_{\perp}$ is an anti invariant submanifold, unless $\bar{M}$ is nearly Sasakian.

Proof: Consider$\xi \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_{\perp})$,then by the structure equation of nearlyLorentzian para-Sasakian manifold

$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = 2g(X,Y)\xi + 4\eta(X)\eta(Y)\xi + \eta(Y)X + \eta(X)Y - 3\eta(Y)\phi X - 3\eta(X)\phi Y + 2g(X,\phi Y)\xi$,

We have

$$\boldsymbol{(\bar{\nabla}_Z\phi)\xi + (\bar{\nabla}_\xi\phi)Z = 2g(Z,\xi)\xi + 4\eta(Z)\eta(\xi)\xi}$$
$$\boldsymbol{+\,\eta(\xi)Z + \eta(Z)\xi - 3\eta(\xi)\phi Z - 3\eta(Z)\phi\xi + 2g(Z,\phi\xi)\xi}$$

$(\bar{\nabla}_Z\phi)\xi + (\bar{\nabla}_\xi\phi)Z = -Z + 3\phi Z$

Using (2.6), weobtain

$$\boldsymbol{-\phi\bar{\nabla}_Z\xi + \bar{\nabla}_\xi\phi Z - \phi\bar{\nabla}_\xi Z = -Z + 3\phi Z}$$

Using (2.11), we obtain,

$$\bar{\nabla}_\xi\phi Z - 2\phi h(Z,\xi) = -Z + 3\phi Z \tag{3.1}$$

Taking the inner product with ϕZ in (3.1) and then using (2.2) and the fact that $\xi \in \Gamma(TN_T)$, we get $\|Z\|^2 = 0$ and hence we conclude that M is invariant, which proves the theorem.

Now, we will discuss the other case, when ξ is tangent to $N_{\perp}$.

Theorem 3.2: Let$\bar{M}$ be a nearly Lorentzian para-Sasakian manifold with a semi symmetric semi metric connection. Then there do not exist warped product submanifolds $M = N_{\perp} \times_f N_T$ such that $N_{\perp}$ is an anti-invariant submanifold tangent to ξ and N_T is an invariant submanifold of$\bar{M}$, unless $\bar{M}$ is nearly Kenmotsu.

Proof: Consider $\xi \in \Gamma(TN_T)$ and $X \in \Gamma(TN_\perp)$, then we have

$$(\overline{\nabla}_X \phi)\xi + (\overline{\nabla}_\xi \phi)X = -X + 3\phi X$$

Using (2.6), we get

$$-\phi(\overline{\nabla}_X \xi) + \overline{\nabla}_\xi \phi X - \phi \overline{\nabla}_\xi X = -X + 3\phi X \tag{3.2}$$

Taking the inner product with X in (3.2) and using (2.2), (2.11), Lemma 2.1 (ii) and the fact that ξ is tangent to $N_\perp$, we obtain $\|X\|^2 = 0$. Thus, we conclude that M is anti-invariant submanifold of a nearly Lorentzian para-Sasakian manifold $\overline{M}$ otherwise $\overline{M}$ is nearly Kenmotsu. This completes the proof.

Now, we will discuss the warped product $M = N_T \times_f N_\perp$ such that the structure vector field ξ is tangent to $N_\perp$.

Theorem 3.3 Let $\overline{M}$ be a nearly Lorentzian para-Sasakian manifold with a semi symmetric semi metric connection. Then there do not exist warped product submanifolds $M = N_T \times_f N_\perp$ such that $N_\perp$ is an anti-invariant submanifold tangent to ξ and N_T is an invariant submanifold of $\overline{M}$.

Proof: If we consider $X \in \Gamma(TN_T)$ and the structure vector field ξ is tangent to $N_\perp$, then by (2.10) we have
$(\overline{\nabla}_X \phi)\xi + (\overline{\nabla}_\xi \phi)X = -X + 3\phi X$.

Using (2.6), we obtain
$-\phi \overline{\nabla}_X \xi + \overline{\nabla}_\xi \phi X - \phi \overline{\nabla}_\xi X = -X + 3\phi X.$

Then by (2.11) and Lemma 2.1 (ii), we derive

$$(\phi X\, lnf)\xi - 2\phi h(X,\xi) + h(\phi X,\xi) = -X + 3\phi X \tag{3.3}$$

Hence, the result is obtained by taking the inner product with ξ in (3.3).

If we consider the structure vector field ξ tangent to N_T for the warped product $M = N_T \times_f N_\perp$, then we prove the following result for later use.

Lemma 3.4: Let $M = N_T \times_f N_\perp$ be a contact CR-warped product submanifold of a nearly Lorentzian para-Sasakian with a semi symmetric semi metric connection of $\overline{M}$ such that N_T and $N_\perp$ are invariant and anti-invariant submanifolds of $\overline{M}$, respectively. Then, we have

(i) $\xi\,(\ln f) = 3,$

(ii) $g(h(X,Y),\phi Z) = 0.$

(iii) $g(h(X,\omega),\phi Z) = g(h(X,Z),\phi\omega) = -\frac{1}{3}\{\eta(X)g(Z,\omega) - (\phi X lnf)g(Z,\omega)\}$

(iv) $3g(h(\xi,Z),\phi\omega) = g(Z,\omega).$

For every $X, Y \epsilon\, \Gamma(TN_T)$ and $Z, \omega\, \epsilon\, \Gamma(TN_\perp)$

Proof: If ξ is tangent to N_T, for any $Z \epsilon \Gamma(TN_\perp)$, we have
$(\overline{\nabla}_\xi \phi)Z + (\overline{\nabla}_Z \phi)\xi = -Z + 3\phi Z.$

Then from (2.6), (2.11) and Lemma 2.1(ii), we get

$$\overline{\nabla}_\xi \phi Z + \phi\left(\nabla_\xi Z + h(Z,\xi)\right) + \phi\left(\nabla_Z \xi + h(Z,\xi)\right) = -Z + 3\phi Z$$

$$2(\xi lnf)\phi Z + 2\phi h(Z,\xi) - \overline{\nabla}_\xi \phi Z = -Z + 3\phi Z. \tag{3.4}$$

Taking the inner product with ϕZ in (3.4) and using (2.2), we derive

$$2(\xi lnf)\|Z\|^2 - g\left(\overline{\nabla}_\xi \phi Z, \phi Z\right) = \|Z\|^2 \tag{3.5}$$

On the other hand, by the property of Riemannian connection, we have

$$\xi g(\phi Z, \phi Z) = 2g\left(\overline{\nabla}_\xi \phi Z, \phi Z\right).$$

Using (2.2) and the property of Riemannian connection, we get

$$g(\overline{\nabla}_\xi Z, Z) = g\left(\overline{\nabla}_\xi \phi Z, \phi Z\right)$$

Using this fact in (3.4) and then from (2.11)and Lemma 2.1 (ii), we deduce that $(\xi \ln f)\|Z\|^2 = \|Z\|^2 \Rightarrow \quad \xi(lnf) = 3$, Forany $Z \epsilon \Gamma(TN_\perp)$, which gives (i)of Lemma (3.1).For the other part of the Lemma, we have

$$(\overline{\nabla}_X \phi)Z + (\overline{\nabla}_z \phi)X = 2g(X, Z)\xi + \eta(Z)X + \quad \eta(X)Z + 4\eta(X)\eta(Z)\xi - 3\eta(Z)\phi X$$
$$-3\eta(X)\phi Z + 2g(X, \phi Z)\xi$$

For any $X \epsilon \Gamma(TN_T)$ and $Z \epsilon \Gamma(TN_\perp)$. Using (2.6), (2.11) & (2.12),we derive

$$\overline{\nabla}_X \phi\, Z - \phi(\overline{\nabla}_X Z) + \overline{\nabla}_z \phi X - \phi \overline{\nabla}_z X = -A_{\phi z}X + \ \nabla_X^\perp \phi Z - \eta(X)\phi Z - \phi \nabla_X Z - \phi h(X, Z) + \nabla_Z \phi X$$
$$+h(\phi X, Z) - \phi \nabla_Z X - \phi h(X, Z)$$
$$= -A_{\phi z}X + \nabla_X^\perp \phi Z - 2(Xlnf)\phi Z + (\phi Xlnf)Z + \ h(\phi X, Z) - 2\phi h(X, Z) - \eta(X)\phi Z$$
$$= \eta(X)Z - 3\eta(X)\phi Z$$
$$\Rightarrow \quad -A_{\phi z}X + \nabla_X^\perp \phi Z - 2(Xlnf)\phi Z + \quad (\phi Xlnf)Z + \ h(\phi X, Z) - 2\phi h(X, Z)$$
$$= \eta(X)Z - 2\eta(X)\phi Z \tag{3.6}$$

Thus, the second part can be obtained by taking the inner product in (3.6) with$Y$, for any $Y\epsilon\, \Gamma(TN_T)$. Again taking the inner product in (3.6) with ω for any $\omega \epsilon \Gamma(TN_\perp)$, we get

$$\eta(X)g(Z, Y) - \ 2\,\eta(X)g(\phi Z, Y) \ = \ -g\left(A_{\phi z}X, Y\right) + \quad g(\nabla_X^\perp \phi Z, Y) - 2(Xlnf)g(\phi Z, Y)$$
$$+(\phi Xlnf)g(Z, Y) + g(h(\phi X, Z), Y) - 2\phi g(h(X, Z), Y)$$
$$0 = -g\left(A_{\phi z}X, Y\right) \Rightarrow g(h(X, Y), \phi Z) = 0 \ \text{ which is (ii) of lemma (3.1).}$$
$$\eta(X)g(Z, \omega) - \eta(X)g(\phi Z, \omega) =$$
$$-g\left(A_{\phi z}X, \omega\right) + \quad g(\nabla_X^\perp \phi Z, \omega) - 2(Xlnf)g(\phi Z, \omega)$$
$$+\phi(X \ln f)g(Z, \omega) + g(h(\,(\phi X, Z), \omega) \ - 2\phi g(h(X, Z), \omega)$$
$$\eta(X)g(Z, \omega) = -g(h(X, \omega), \phi Z) + (\phi Xlnf)g(Z, \omega)$$
$$-2g(h(X, Z), \phi\omega) \tag{3.7}$$

By polarization identity, we get

$$\eta(X)g(Z, \omega) = - \ g(h(X, Z), \phi\omega) + (\phi X \ln f)g(Z, \omega)$$
$$- 2g(h(X, \omega), \phi Z) \tag{3.8}$$

Then from (3.7) and (3.8), we obtain

$$- \ g(h(X, \omega), \phi Z) + (\phi X \ln f)g(Z, \omega) - \quad 2g(h(X, Z), \phi\omega)$$
$$= \ - \ g(h(X, Z), \phi\omega) + \quad (\phi X \ln f)g(Z, \omega) - \ 2g(h(X, \omega), \phi Z)$$

Or

$$2g(h(X, \omega), \quad \phi Z) - g(h(X, \omega), \quad \phi Z) =$$
$$-g(h(X, Z), \phi\omega) + 2g(h(X, Z), \phi\omega)$$

$$g(h(X,\omega),\ \phi Z) = g(h(X,Z),\phi\omega) \tag{3.9}$$

Which is the first equality of lemma 3.4 (iii). Using (3.9) either in(3.7) or in (3.8),

We get the second equality oflemma 3.4 (iii).

Now for the last part, replacing X by ξ in the third part of this lemma, we obtained

$$g(h(X,\omega)\,,\phi Z) = g(h(X,Z),\phi\omega) = -\frac{1}{3}\{\eta(X)g(Z,\omega) - (\phi X \ln f)g(Z,\omega)\}$$

Using above relation we get

$$3g(h(X,Z)\,,\phi\omega) - (\phi\xi \ln f)g(Z,\omega) = -\eta(X)g(Z,\omega)$$

Put $X = \xi$, we get

$$3g(h(\xi,Z),\phi\omega) - (\phi\xi \ln f)g(Z,\omega) = -\eta(\xi)g(Z,\xi)$$
$$3g(h(\xi,Z),\phi\omega) = g(Z,\omega)$$

Which is lemma 3.4 (iv).Thus we have proved all part of lemma 3.4

Now, we have the following characterization theorem.

Theorem 3.5: Let M be a contact CR-submanifold of a nearly Lorentz para-Sasakian manifoldwith a semi symmetricsemi metric connection$\overline{M}$ with integrable invariant and anti-invariant distribution $D \oplus\langle\xi\rangle$and $D^{\perp}$. Then M is locally a contact CR-warped product if any only if the shape operator of M satisfies

$$A_{\phi\omega}X = \frac{1}{3}\{(\phi X\mu)\omega - \eta(X)\omega\}\ \forall\, X\ \in \Gamma(D\oplus\langle\xi\rangle),$$
$$\forall\ \omega\ \in \Gamma(D^{\perp}) \tag{3.10}$$

For some smooth function μ on M satisfying V $(\mu) = 0\ for\ every\ V\ \epsilon\Gamma(D^{\perp})$.

Proof: Direct part follows from the Lemma 3.4 (iii). For converse, suppose M is contact CR-submanifold satisfying (3.10), then we have $g(h(X,Y),\phi\omega) = g\big(A_{\phi\omega}X,Y\big) = 0$, for any $X,Y\ \in \Gamma(D\oplus\langle\xi\rangle$ and $\omega\ \in \Gamma(D^{\perp})$. Using (2.2) and (2.11), we get

$g(\overline{\nabla}_X Y,\phi\omega) = -g(\phi\overline{\nabla}_X Y,\omega) = 0$.Then from (2.6), we obtain

$$g\big((\overline{\nabla}_X\phi)Y,\omega\big) = g(\overline{\nabla}_X\phi Y,\omega). \tag{3.11}$$

Similarly, we have

$$g\big((\overline{\nabla}_Y\phi)X,\omega\big) = g(\overline{\nabla}_Y\phi X,\omega) \tag{3.12}$$

Then from (3.11)and (3.12), we derive

$$g\big((\overline{\nabla}_X\phi)Y + (\overline{\nabla}_Y\phi)X,\omega\big) = g(\overline{\nabla}_X\phi Y + \overline{\nabla}_Y\phi X,\omega). \tag{3.13}$$

Using (2.10) and the fact that ξ is tangent to N_T, then by orthogonality of two distributions, we obtain

$$g(\overline{\nabla}_X\phi Y + \overline{\nabla}_Y\phi X,\omega) = 0 \tag{3.14}$$

This means that $\nabla_X\phi Y + \nabla_Y\phi X\ \epsilon\ \Gamma(D\oplus\langle\xi\rangle$, for any $X,Y\ \epsilon\ \Gamma\ (D\oplus\langle\xi\rangle)$, that is $D\oplus\langle\xi\rangle$ is integrable and its leaves are totally geodesic in M. So for as the anti-invariant distribution $D^{\perp}$ is connected,it is integrable on M (cf.[3], Theorem 3.4). Moreover, for any $X \in \Gamma(D\oplus\xi)$ and $Z, \omega \in \Gamma(D^{\perp})$ and h^{*} be the second fundamental form of $N_{\perp}$in M. We have

$g(h^{*}(Z,\omega),\phi X) = g(\nabla_{Z}\omega,\phi X)$, Using (2.2), (2.6) and (2.11), we get

$g(h^{*}(Z,\omega),\phi X) = g(\overline{\nabla}_{Z}\phi)\omega, X) - g(\overline{\nabla}_{Z}\phi\omega, X)$.

Then from (2.12) and (2.13), we get

$$g(h^{*}(Z,\omega),\phi X) = g\big((\overline{\nabla}_{Z}\phi)\omega, X\big) + g\big(A_{\phi\omega}X, Z\big) \tag{3.15}$$

Using (3.10), we derive

$$g(h^{*}(Z,\omega),\phi X) = g\big((\overline{\nabla}_{Z}\phi)\omega, X\big)$$
$$+\frac{1}{3}\{\,(\phi X)\mu) - \ \eta(X)\}g(Z,\omega) \tag{3.16}$$

Similarly, we obtain

$$g(h^{*}(Z,\omega),\phi X) = g\big((\overline{\nabla}_{\omega}\phi)Z, X\big)$$
$$+\frac{1}{3}\{\,(\phi X)\mu) - \eta(X)\}g(Z,\omega) \tag{3.17}$$

Now adding (3.16) and (3.17), we get

$$2g(h^{*}(Z,\omega),\phi X) = g\big((\overline{\nabla}_{Z}\phi)\omega + (\overline{\nabla}_{\omega}\phi)Z, X\big) + \frac{2}{3}\{(\phi X)\mu - \eta(X)\}g(Z,\omega) \tag{3.18}$$

Replacing $X = Z, \quad Y = \ \omega$ in equation (2.10), we get

$(\overline{\nabla}_{Z}\phi)\omega + (\overline{\nabla}_{\omega}\phi)Z = 2g(Z,\omega)\xi$ With the fact that ξ is tangent to $N_{T}$and using it in (3.18), we obtain

$$g(h^{*}(Z,\omega),\phi X) = g(Z,\omega)g(\xi, X)$$
$$+\frac{1}{3}\{(\phi X)\mu - \eta(X)\}g(Z,\omega) \tag{3.19}$$

$$g(h^{*}(Z,\omega),\phi X) = \frac{1}{3}\{(\phi X)\mu g(Z,\omega)\} \tag{3.20}$$

Using (2.14), we derive

$$g(h^{*}(Z,\omega),\phi X) = \frac{1}{3}g(\nabla\mu,\phi X)g(Z,\omega) \tag{3.21}$$

From the last relation, we obtain that

$$g(h^{*}(Z,\omega),\phi X) = \frac{1}{3}(\nabla\mu)g(Z,\omega). \tag{3.22}$$

The above relation shows that the leaves of $D^{\perp}$ are totally umbilical in M with mean curvature vector$\nabla\mu$. Moreover, the condition $V\mu$=0, for any $V \in \Gamma(D^{\perp})$ implies that the leaves of $D^{\perp}$ are extrinsic spheres in M, that is the integral manifold $N_{\perp}$ of $D^{\perp}$ is umbilical and its mean curvature vector field is non-zero and parallel along $N_{\perp}$. Hence by a result of [11]M is locally a warped product $M = N_{T} \times_{f} N_{\perp}$, where N_{T} and $N_{\perp}$ denote the integral manifolds of the distributions$D\oplus\langle\xi\rangle$and$D^{\perp}$, respectively and $f$is the warping function. Thus, the theorem is proved completely.

4. Inequality for Contact CR-Warped Products

In the following section, we obtained a general sharp inequality for the length of second fundamental form of warped product submanifold.

Theorem 4.1: Let $M = N_T \times_f N_\perp$be a contact CR-warped product submanifold of a nearly Lorentzian para-Sasakian manifold with semi symmetric semi metric connection$\overline{M}$ such that N_T is an invariant submanifold tangent to ξ and $N_\perp$ an anti invariant submanifold of $\overline{M}$.Then, we have

(i) The second fundamental form of M satisfyies the inequality

$$\|h\|^2 \geq \frac{2}{9}s + \frac{2}{9}s\,\|\nabla ln f\|^2 \tag{4.1}$$

Where s is thedimension of $N_\perp$ and $\nabla \ln f$ is the gradient of $\ln f$.

(ii) If the equality sign of (4.1) hold identically, then N_T is the totally geodesic submanifold and $N_\perp$ is the totally umbilical submanifold of $\overline{M}$. Morever, M is a minimal submanifold in$\overline{M}$.

Proof: Let $\overline{M}$ be a $(2n+1)$-dimantional nearly Lorentzian para-Sasakian manifold with a semi symmetricsemi metricconnection and $M = N_T \times_f N_\perp$be an m-dimentional contact CR-warped product submanifold of $\overline{M}$. Let us considerdim $N_T = 2p+1$, and $dim N_\perp = s$ then$m = 2p+1+s$.

Let$\{e_1 \dots, e_p; \phi e_1 = e_{p+1}, \dots, \phi e_p = e_{2p}, e_{2p+1} = \xi\}$ and $\{e_{(2p+1)+1}, \dots\dots, e_m\}$be the local orthogonal frames on $N_T$and $N_\perp$ respectively. Then the orthogonal frames in the normal bundle$T^\perp M$of $\phi D^\perp$ and υ are $\{\phi e_{(2p+1)+1}, \dots, \phi e_m\}$and $\{e_{m+s+1}, \dots, e_{2n+1}\}$ respectively. Then the length of second fundamental form h is defined as

$$\|h\|^2 = \sum_{r=m+1}^{2n+1} \sum_{i,j=1}^{m} g(h(e_i, e_j), e_r)^2 \tag{4.2}$$

For the assumed frames, the above equation can be written as

$$\|h\|^2 = \sum_{r=m+1}^{m+s} \sum_{i,j=1}^{m} g(h(e_i, e_j), e_r)^2 + \sum_{r=m+s+1}^{2n+1} \sum_{i,j=1}^{m} g(h(e_i, e_j), e_r)^2. \tag{4.3}$$

The first term in the right hand side of the above equality is the $\phi D^\perp$-component and the term is υ-component. If we have only the $\phi D^\perp$-component, then we have

$$\|h\|^2 \geq \sum_{r=m+1}^{m+s} \sum_{i,j=1}^{m} g(h(e_i, e_j), e_r)^2 \tag{4.4}$$

For the given frame of $\phi D^\perp$, the above equation will be

$$\|h\|^2 \geq \sum_{k=(2p+1)+1}^{m} \sum_{i,j=1}^{m} g(h(e_i, e_j), \phi e_k)^2$$

Let us decompose the above equation in terms of the component of $h(D,D), h(D,D^\perp)$ and $h(D^\perp, D^\perp)$, then we have

$$\|h\|^2 \geq \sum_{k=2p+2}^{m} \sum_{i,j=1}^{2p+1} g(h(e_i, e_j), \phi e_k)^2$$

$$+2 \sum_{k=2p+2}^{m} \sum_{i=1}^{2p+1} \sum_{j=2p+2}^{m} g(h(e_i, e_j), \phi e_k)^2$$

$$+ \sum_{k=2p+2}^{m} \sum_{i,j=2p+2}^{m} g(h(e_i, e_j), \phi e_k)^2. \tag{4.5}$$

By Lemma 3.4 (ii), the first term of the right hand side of (4.5) is identically zero and we will compute the next term and will left the last term

$$\|h\|^2 \geq 2\sum_{k=2p+2}^{m}\sum_{i=1}^{2p+1}\sum_{j=2p+2}^{m} g(h(e_i,e_j),\phi e_k)^2.$$

As $j,k = 2p+2, \ldots\ldots, m,$ then the above equation can be written for one summation as

$$\|h\|^2 \geq 2\sum_{i=1}^{2p+1}\sum_{j,k=2p+2}^{m} g(h(e_i,e_j),\phi e_k)^2.$$

Now using the lemma 3.4 (iii), the above inequality become

$$\|h\|^2 \quad \geq 2\textstyle\sum_{i=1}^{2p+1}\sum_{j,k=2p+2}^{m}[\frac{1}{3}(\phi e_i lnf)g(e_j,e_k) - g(e_j,e_k)\eta(e_k)]^2. \tag{4.6}$$

$$\|h\|^2 \geq \frac{2}{9}\sum_{i=1}^{2p+1}\sum_{j,k=2p+2}^{m}(\phi e_i lnf)^2 g(e_j\,,e_k)^2$$

$$+\frac{2}{9}\sum_{i=1}^{2p+1}\sum_{j,k=2p+2}^{m}\eta(e_i)^2\, g(e_j,e_k)^2$$

$$-\tfrac{4}{9}\textstyle\sum_{i=1}^{2p+1}\sum_{j,k=2p+2}^{m}(\phi e_i\, lnf)\eta(e_i)g(e_j\,,e_k)g(e_j\,,e_k) \tag{4.7}$$

The last term of (4.7) is identically zero for the given frames. Thus the above relation gives

$$\|h\|^2 \quad \geq \tfrac{2}{9}\textstyle\sum_{i=1}^{2p+1}\sum_{j,k=2p+2}^{m}(\phi e_i lnf)^2 g(e_j\,,e_k)^2 + \tfrac{2}{9}s \tag{4.8}$$

On the other hand from (2.15), we have

$$\|\nabla lnf\|^2 = \sum_{i=1}^{n}(e_i(f))^2 + \sum_{i=1}^{n}(\phi e_i lnf)^2 + (\xi lnf)^2 \tag{4.9}$$

Now, the equation (4.8) can be modified as

$$\|h\|^2 \geq \frac{2}{9}s + \frac{2}{9}\sum_{i=1}^{2p+1}\sum_{j,k=2p+2}^{m}(\phi e_i lnf)^2 g(e_j\,,e_k)^2$$

$$+\frac{2}{9}\sum_{j,k=2p+2}^{m}(\xi lnf)^2 g(e_j,e_k)^2$$

$$-\frac{2}{9}\sum_{j,k=2p+1}^{m}(\xi lnf)^2\, g(e_j\,,e_k)^2$$

$$\|h\|^2 \geq \frac{2}{9}s + \frac{2}{9}\sum_{i=1}^{2p}\sum_{j,k=2p+1}^{m}(\phi e_i lnf)^2 g(e_j\,,e_k)^2$$

$$+\frac{2}{9}\sum_{j,k=2p+1}^{m}(\xi lnf)^2 g(e_j,e_k)^2$$

$$-\frac{2}{9}\sum_{j,k=2p+1}^{m}(\xi lnf)\,g(e_j,e_k)^2$$

$$\|h\|^2 \geq \frac{2}{s}s-\frac{2}{9}\sum_{j,k=2p+2}^{m}(\xi lnf)^2 g\left(e_j,e_k\right)^2$$

$$-\frac{2}{9}\sum_{i=1}^{2p+1}\sum_{j,k=2p+2}^{m}(\xi lnf)^2 g\left(e_j\,,e_k\right)^2$$

$$+\frac{2}{9}\sum_{i=1}^{p}\sum_{j,k=2p+2}^{m}(\phi e_i lnf)^2\;g\left(e_j,e_k\right)^2$$

$$+\frac{2}{9}\sum_{j,k=2p+2}^{m}(\xi lnf)\,g(e_j,e_k)^2$$

Or

$$\|h\|^2 \geq \frac{2}{9}s-\frac{2}{9}\sum_{j,k=2p+2}^{m}(\xi lnf)^2\,g\left(e_j,e_k\right)^2$$

$$+\frac{2}{9}\sum_{i=1}^{p}\sum_{j,k=2p+2}^{m}(\phi e_i lnf)^2\;g(e_j,e_k)^2$$

$$-\frac{2}{9}\sum_{i=1}^{p}\sum_{j,k=2p+2}^{m}(e_i lnf)^2\,g\left(e_j,e_k\right)^2$$

$+\frac{2}{9}\sum_{i=1}^{p}\sum_{2p+2}^{m}\|\nabla lnf\|^2\,g(e_j,e_k)^2(\because\ \phi\xi lnf\,=0)$

Therefore, using Lemma 3.4 (i) and (4.9), we arrive at

$$\|h\|^2 \geq\ \frac{2}{9}s+\frac{2}{9}s\|\nabla lnf\|^2$$

This is the inequality (4.1). Let h^* be the second fundamental form of $N_\perp$ in M, then from (3.22), we have $h^*(Z,\omega)=g(Z,\omega)\nabla\ln f$(4.10)

For any $Z,W\in\Gamma(D^\perp)$. Now assume that the equality case of (4.1) holds identically. Then from (4.3), (4.5) and (4.7), we obtain
$h(D,D)=0,\ \ h(D^\perp,D^\perp)=0,\ \ h(D,D^\perp)\subset\phi D^\perp.$ (4.11)

Since N_T is a totally geodesic submanifold in M(by Lemma 2.1 (i), using this fact with the first condition in (4.11) implies that N_T is totally geodesic in $\bar{M}$. on the other hand, by direct calculation same as in the proof of Theorem 3.5, we deduce that $N_\perp$ is totally umbilical in M. therefore, the second condition of (4.11) with (4.10) implies that $N_\perp$ is totally umbilical in $\bar{M}$. Moreover, all three conditions of (4.11) imply that M is minimal submanifold of$\bar{M}$. Hence we have proved the theorem.

5. Author's Contributions

The first author contributed in conception, studied, the problems and prepared the draft manuscript. Second and third authors contributed in literature review and revised the whole paper. All authors have reviewed andedited the manuscript and approved the final version of the manuscript.

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References

[1] N. K. Agrawal, S. Rahman, "Characterization of canonical semi symmetric semi-metricconnection on transversal hyperSu faces of Lorentzian para Sasa-kian manifolds",International Journal of Emerging Technology and Advanced Engineering., Vol 6(10), pp.153–159, 2016.

[2] A. Ahmad, S. Rahman, "A note on transversal hyper Surfaces of Lorentzian para-Sasakianmanifolds with a Semi-symmetric non-metric connection," Journal of Tensor Society. Vol8, pp. 53–63, 2014

[3] P. Alegre, "CR-submanifolds of Lorentzian manifolds," Int. Elect. J. Geom. Vol 2, pp. 95–103, 2009

[4] K.Arslan, R. Ezentas, I. Mihai, C. Murathan, "Contact CR-warped product submanifolds in Kenmotsu space forms," J.Korean Math. Soc. Vol 42, pp. 1101–m1110, 2005

[5] R.L. Bishop, B. O'Neill "Manifolds of negative curvature," Trans. Amer. Math. Soc. 145, pp. 1–49, 1969

[6] V. Bonanzinga, K. Matsumoto, "Warped product CR-submanifolds in locally conformal Kaehler manifolds" Periodica Math. Hungar., 48, pp. 207–221, 2004

[7] B.Y. Chen, "Geometry of warped product CR-submanifolds in Kaehler manifoldsI," Monatsh. Math., Vol.133, pp 177–195, 2001.

[8] B.Y. Chen,"Geometry of warped product CR-subman infolds in Kaehler manifolds II," Monatsh. Math., Vol 134, pp. 103–119, 2001

[9] U.C. De, K. Matsumoto, A.A. Sheikh, "On Lorentzian para-Sasakian manifolds,"RendicontidelSeminario Matematico di Messina, Series II, Supplemento al n. Vol 3, pp. 149–158, 1999.

[10] I. Hasegawa, I. Mihai,"Contact CR-warped product submanifolds in Sasakian manifolds," Geom. Dedicata, Vol. 102, pp. 143– 150, 2003.

[11] S. Hiepko,"Eine inner kennzeichungder verzerrten produkte," Math. Ann., 241, pp. 209–215, 1979.

[12] K. Matsumoto, "On Lorentzian para contact manifolds," Bull. of Yamagata Univ., Nat. Sci., Vol 12, pp. 151–156, 1989.

[13] I. Mihai, "Contact CR-warped product submanifolds in Sasakian space forms," Geom.Dedicata, 109, pp. 165–173, 2004.

[14] I. Mihai, R. Rosca, "On Lorentzian P-Sasakian Manifolds," Classical Analysis, World Scientific Publ., pp. 155–169, 1992.

[15] S. Rahman,"Transversal hypersurfaces of a Lorentzian para-Sasakian manifold with semi symmetric metric connection,"Jounal of Qafqaz University- Mathematics and Computer Science, Vol.1(2), pp. 137–144, 2013.

[16] S. Rahman, "Characterization of quarter symmetric non-metric connection on transversal hypersurfaces of Lorentzian para Sasakian manifolds," Journal of Tensor Society, Vol 8, pp. 65–75, 2014.

[17] S. Rahman,"Contact CR-warped product submain folds of nearly Lorentzian para-Sasakian manifold," Turkish Journal of Mathematics & Computer Science, Vol 7, pp. 40-47, 2017.

[18] S. Rahman,"Characterization of quarter symmetric semi metric connection on transversal hypersurfaces of Lorentzian para-Sasakian manifolds," Journal of Qafqaz University- Mathematics and Computer Science, Vol. 3(1), pp. 09–17, 2015.

[19] S. Rahman,J.B. Jun, A. Ahmad, “On semi-invariant submanifolds of a nearly Lorentzian para-Sasakian manifold,” FarEast Journal of Mathematical Sciences, Vol. 96(6), pp. 709–724, 2015.

[20] B. Sahin, “Non-existence of warped product semi slant submanifold of Kaehler manifolds,”Geom. Deticata. 117, pp. 195 – 202, 2006

[21] S. Rahman, M. S. Khan and A Horaira, “Warped Product Semi Slant Submanifold of Nearly Quasi Sasakian Manifold”. Boson Journal of Modern Physics (BJMP) Vol. 5 No. 1, pp.443-453(2019)

[22] S. Rahman, M. S. Khan and A. Horaira, “Contact CR-warped product submanifolds of nearly Quasi-Sasakian manifold”, Konuralp Journal of Mathematics, Vol. 7 No. 1, pp.140-145, (2019)

[23] S. Rahman, M.S. Khan and A.Horaira,“CR-submanifolds of a nearly trans-hyperbolic Sasakian manifold with a quarter symmetric metric connection”.,Journal-Proceedings of the Institute of Applied Mathematics, Vol.8 No.1, pp.24-34, (2019).