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Enhancing Sustainability in Fish Harvesting: Analyzing Constant and Proportional Growth Models

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Abstract
This paper aims to explore the population dynamics of fish using a logistic growth model including harvesting effects. Specifically, this study looks at two different harvesting strategies, known as constant and proportional harvesting. The aim is to establish the best patterns for growth and reproduction in fish populations at these different harvesting rates. By analysing the results of constant versus proportional harvesting, this research aims to find out whether one is more effective than the other in striking a balance between sustainable management of fish populations and economic viability. In conclusion, through such an analysis it becomes possible to make recommendations on how fisheries should be managed to achieve sustainability while optimizing resource use efficiency. This research will have far-reaching implications that will ensure policies are well made and best practices are taken into consideration by managing fisheries thus contributing towards sustainability and productivity of fish populations in the long run.

Key words: Logistic Growth Model, Constant Harvesting, Proportional Harvesting, Fishing effort, Production Function, Linear Complementarity Problem.

1. Introduction

A Population is a group of organisms of the same species (fishes, birds etc.) that live in a particular area. The number of organisms in a population changes over time because of births, deaths, emigration, immigration and some outside factors. Of course, births and immigration for example increase the size of the population, on the other hand deaths and emigration for example decrease its size. The increase in the number of organisms in a population is mentioned as population growth. There are factors that can help population grow and other than can slow down populations from growing. Factors that limit population growth are called limiting factors. In population dynamics, the growth of a population can be described if the functional behavior of the rate of growth is known. Arne[1] ,Auger[2] studied single species population models in several forms,, involving ordinary differential equations, delay differential equations and the discrete maps. Most often, Logistic equation is used with an assumption that reducing density results in the same or higher per capita growth which implies that populations are resilient and recover rapidly when factors causing the decline are removed. Fish populations exhibit social co-operation behaviourism which provides individuals a greater chance of survival and reproduction as the density increases, whereas when reduced to low densities, they experience reduced survival and reproduction rates. Courchamp[3] and Elaydi[4] made intensive study on the existence of the maximum sustainable yield (MSY), which is the maximum proportion that can be removed from the stock over time without causing population decline below the optimum level.

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Benjamin Gompertz an English Actuary and mathematician was the first person to prose a mathematical model in the year 1820. This model was used for the first time by Verhulst[5] for cumulative growth of a product, especially human population. The experimental date have been reasonable fir with this model by Pearl[6] in the fruit fly population. Feller[7] pointed out that almost any data any for populations that increase to an asymptotic level will fit to the logistic model to some degree. Many alternative form of the growth function have been investigated by many workers. Studied the model with generalized function.Chaudhari and Pradhan[8] studied Harvesting of a prey-predator fishery with low predator density. In the fisheries industry this model was studied by Holling[9], Larkin [10], and Clark[11] made intensive study of this model for the application of the commercial fishery industry and forest industry. Logistic model is frequently used in evaluating and planning new ventures. This research will have far-reaching implications that will ensure policies are well made and best practices are taken into consideration by managing fisheries thus contributing towards sustainability and productivity of fish populations in the long run application of the logistic model in renewable resources especially in the fishery industry.

2. Logistic Growth Model:

Suppose that in a certain population size, both the birth rate b and mortality rate m is proportional to the population size n(t)

$$\frac{dn(t)}{dt} = rn(t) \qquad (1)$$

where, $r = (b - m)$ net proportional growth rate of population

$$\frac{dn(t)}{n(t)} = r\,dt$$

$$n(t) = ce^{rt}$$

$$n(t) = n_o e^{rt}, \quad n_0 = n(0) \qquad (2)$$

n(t) grows exponentially t infinite if r>0 and decreases exponentially to zero if r<0. To remove unrestricted growth Verhulst[5] considered that a stable population would have a saturation level characteristic of the environment. To achieve this the exponential model was augmented by a multiplicative factor $\left(1 - \frac{n(t)}{N}\right)$, which represents the fractional deficiency of the current size from the saturation level K .

In LotKa analysis of the logistic growth concept the rate of population growth $\frac{dn}{dt}$ at any moment t, in a function of the population size of that moment n(t),

$$\frac{dn}{dt} = f(n)$$

Since a zero population has zero growth, n= 0 is an algebraic root of the yet unknown function f(n). By Taylor's series, near n = 0 and setting f(0)=0

$$f(n) = f(0) + nf'(0) + \frac{n^2}{\boxed{2}} f''(0) + \ldots\ldots\ldots\ldots$$

$$f(n) = n\left[f'(0) + \frac{n}{2} f''(0)\right]$$

By setting $f'(0)$= r, $f''(0) = -\frac{2r}{K}$, where K is the carrying capacity, are is led to Verhulst Logistic equation

$$\frac{dn}{dt} = rn\left[1 - \frac{n}{K}\right] \qquad (3)$$

The Verhulst logistic equation is also referred to in the literature as the Verhulst-pearl equation after Verhulst[5], who first derived the curve, and Pearl[6], who used the curve to approximate population growth in the united states in 1920.

Soultion of equation (3)

$$n(t) = \frac{K}{1+\left(\frac{K}{n_o}-1\right)e^{-rt}} \tag{4}$$

The three key features of the logistic growth are :

(i) $\lim_{t\to\infty} n(t) = K$, the population will ultimately reach its carrying capacity.

(ii) The relative growth rate $\frac{1}{n}\frac{dn}{dt}$ declines linearly with increasing population size.

(iii) The population at the inflection point (Where growth rate is maximum), is exactly half the carrying capacity, $n_{inf} = \frac{k}{2}$

For r > 0, the resulting growth curve has a sigmoidal shape and from (2), is asymptotic to the carrying capacity.

When r < 0 and a reduction in the growth rate per capita is present, the growth curve is asymptotic to zero leading to population extinction.

In the trivial case of no intrinsic growth rate, r = 0, the population remains static at the initial value of n_0.

Population biologists and ecologists are interested mainly in the case where r>0.

3. Fishing harvesting Model

Assume that fish harvesting is started in the environment Sethi,Branch and Waston[12] . The modelling problem is how to maximize the sustainability of the yield by determining the population, growth dynamics so as to fix the harvesting rate that keeps the problem at its maximize growth rate . A control variable of every fishery management is the fishing effort, which is defined as a measure of the intensity of fishing operations. The model (1) takes the form

$$\frac{dn}{dt} = rn\left(1-\frac{n}{K}\right) - h(t)$$

$$\frac{dn}{dt} = F(n) - h(t)\ ,\ F(n) = rn(1-\frac{n}{K}) \tag{6}$$

Constant Harvesting Model

If rate of harvesting h(t)=h(constant) then equation (6) becomes

$$\frac{dn}{dt} = F(n) - h \tag{7}$$

When $h < \max F(n) = \frac{1}{4}rK$ equation (7) has two equilibria, n_1, n_2 when $\frac{dn}{dt} > 0$.n lies between n_1 and n_2 while n<0 ,n lies elsewhere.

Inspite of various limitation the model given by equation (7) provides several prediction concerning the harvesting of renewable resources. If $h_{MSY} =$ max F(n), then a larger harvest rate will lead to the depletion of the population. When the natural equilibrium level is at $n = \frac{K}{2}$ but not at n=K.

Then there is no sustainable yield at the population level n= K

When r =0.75, K=48000 and h is constant. To determine the equilibrium points for h, we have

$$0.75\,n - \frac{0.75n^2}{48000} - h = 0$$

$$n = \frac{0.75 \pm \sqrt{(0.75)^2 - 4h(0.000015625)}}{2(0.000015625)}$$

For the maximum sustainable harvesting rate. We let the expression under the square root sign equal zero as

$$(0.75)^2 - 4(0.000015625)h = 0$$

$$\text{or} \quad 0\,.5625 = 0.0000625h$$

$$\text{or} \quad h = \frac{0.5625}{0.0000625} = 9000$$

The value h = 9000

The maximum sustainable (MSY) or the total allowance catch that can be harvested for the stock or biomass.

The value h=9000 is called Bifurcation point and at thus put we considered three value of harvesting

$$h = 9000, h > 9000\ and\ h < 9000$$

$$\text{For} \quad h = 9000$$

$$n = \frac{0.75 \pm \sqrt{(0.75)^2 - 4(0.00015625)9000}}{2(0.000015625)}$$

$$= 24000$$

We know one equilibrium put for no long than 24000 the equilibrium will decrease and approach to 24000 likewise for no less than 24000 the equilibrium with lead to extinction.

Proportional Harvesting Model

The model starts from the logistic equation and assumes a fishing level per unit of time proportional to the fish stock. Model (6) becomes

$$\frac{dn}{dt} = F(n) - h(t)n \tag{9 }$$

Where h(t) is the harvesting yield per unit time and h(t) is positive constant such that the measure of the effort expended. The algebraic solution is complain and large to interpret, then we again turn to the geometric analysis of the model. The solution of the equation (9)

$$rn^*\left(1 - \frac{n^*}{K}\right) = h(t)\,n^*$$

$$n_1^* = 0, \qquad n_2^* = \frac{(r-h)K}{r}$$

The extinction fixed point $n^* = 0$ is unstable for $h < r$. As h increases, the large equilibrium shrints put it remains stable for $h < r$.

$n = \frac{(r-h)k}{r}$ which is a non-trivial equilibrium , If the point $r > h$.

Clearly for $h > r, n = \frac{k(r-h)}{r} < 0$

Which shows that 0 is the only equilibrium point. The non-trivial equilibrium point is an asymptotic growth value of the harvesting fish model. Since $K\frac{(r-h)}{r} \leq K$ for $r-h>0$ implies that the asymptotic values of the harvesting fish population are lower than the non-harvesting fish population.

If $h<r$ the maximum sustained yield (MSY) denoted by h_{MSY} is obtained by the product of the effort y and the non-trivial equilibrium point $=\frac{K(r-h)}{r}$. In other words

$$h_{MSY}=\frac{hK(r-h)}{r}=hK-\frac{h^2K}{r}$$

$$\frac{dh_{MSY}}{dh}=K-\frac{2hK}{r}$$

$$\frac{dh_{MSY}}{dh}=0, \qquad K-\frac{2Kh}{r}=0$$

$$\therefore h=\frac{r}{2}$$

$$\frac{d^2h_{MSY}}{dh^2}=-\frac{2K}{r}<0$$

which shows that h_{MSY} has a maximum value at $\frac{r}{2}$. Therefore

$$h_{MSY}=\frac{rn}{2}=\frac{rK}{4} \tag{10}$$

This shows that the maximum yield is obtained of the equilibrium point n reader half of its carrying capacity k or the maximum yield if the measure of the effort expanded h is half of the growth rate r.

If the carrying capacity K=48000, intrinsic growth rate r=0.75 for maximum sustainable yield the target population size is $\frac{K}{2}$.

Therefore n= 24000

Then

$$\frac{dn}{dt}=rn\left(1-\frac{n}{K}\right)-h(t)n \tag{11}$$

Substituting the value we get

$$\frac{dn}{dt}=(0.75)24000\left(1-\frac{24000}{48000}\right)-24000h$$

For the equilibrium $,\frac{dn}{dt}=0$

h = 0.375

the optimal harvest rate is 0.375 or 0.4 (approximate). This means 40% of the fish population should be harvest annually to maintain the population at a sustainable level around half the carrying capacity.

From the above the proportional harvests a fixed proportion of the population annually and costant harvests a fixed number of fish annualy both methods aim at a sustainable level of annual 50% of capacity.

4. Application of the Model for the Fishing Effort:

Gordon [13] developed the mathematical model to study open access fishery industry. This model can be applied to many biological resource industry. This model is sometimes referred as the Gordon Schafer model, because the logistic growth model has been used extensively by fisheries biologist M. Schaefer[14]. Let Y denotes sustainable yield, e the level of fishing effort, then Y depends on e . If we assume a constant price p per unit of harvested biomass then, T_R = pY (e), represents the total sustainable revenue resulting from the effort e. If T_C is fishing cost then T_C is proportional to the effort i.e. T_C = Ce, where C is constant.

Sustainable economic rent =T_R-T_C=pY(e)-Ce.

According to the Gordon's principle[13], fishing effort tends to reach on equilibrium (bionomic equilibrium) at the level e = e_∞ at which T_R equals to T_C that is economic rent is completely dissipated. This conclusion can be justified based on two arguments.

(a) If e > e_∞ than fishing industry loose money and will stop the industry.

(b) If e < e_∞ then the fishery industry would earn profit.

5. Production Function for the Resource Industry:

If F(n) denotes the natural growth rate of a given fish population and h(t), the rate of harvesting then the rate of change of population level is

$$\frac{dn}{dt} = F(n) - h(t) \qquad t \geq 0$$

The harvest rate h(t) is assumed to depend on two quantities, current size of stock
n (t) and the rate of harvesting effort e(t)
i.e h=Q(e,n)=qne

where q be a positive constsnt called catchability ,Makwata[15] are considerd the non linear variation in market prices.

The function Q(e,n) which relate e and n is termed as production function Srivastava and Gupta[16]. The form of Q is taken as,

$$Q(e,n) = ae^{\propto}n^{\beta}.$$

This is the log linear relationship as logQ depends on loge and logn and $\propto and\ \beta$ are calculated by linear regression. For the mathematical reason of convenience, we take h as a linear function h=q(e ,n) =g (n).E

This model is easier to handle than a non-linear model g(n) is non decreasing function. If p price of harvested resource is fixed and the cost C of a unit effort is also constant then the net economic revenue produced by an input effort e Δt is rΔt which is equal to [p-C(n)] h. Δ (t) when C(n) =$\frac{C}{g(n)}$

The sole objective of an industry is to maximize revenue from the exploitation of resources.
Our goal is to calculate the effort e that maximize the fisherman χ

$$\chi\, e = ph - ce \tag{12}$$

Where p is the price of the fish population and the total is ph= pqen.
at equilibrium we will have

$$n = K\ 1 - \frac{q}{r}e$$

then

$$\chi e = -K\frac{pq^2}{r}e^2 + pqK - ce$$

According to χ is a second-order function with $-K\frac{pq^2}{r} < 0$, then χhas a unique maximum e* given by

$$e^* = \frac{r}{2}\left(\frac{1}{q} - \frac{c}{Kpq^2}\right)$$

The interest in the second part of this section concerns the study of a bioeconomic problem of one species exploited by two fishermen following the equation where $h_i = qe_in$ for i=1,2

Our objective is to find $e, e_1, e_2,$ which is maximize the profit χ, χ_1, χ_2 of the two fishermen.
At the equilibrium, we will have

$$n = K\ 1 - \frac{1}{r}q\ e_1 + e_2$$

Then the first fisherman must solve the problem (1)

$$max\chi_1\ e = -\frac{K}{r}pq^2e_1^2 + \frac{K}{r}\left(rpq - \frac{c_1}{K}r - pq^2e_2\right)e_1$$

$$subject\ to\ \begin{cases} q_1e_1 + q_2e_2 \leq r \\ e_2 > 0 \\ e_1\ is\ given \end{cases}$$

And the first fisherman must solve the problem (2)

$$max\chi_1\ e = -\frac{K}{r}pq^2e_2^2 + \frac{K}{r}\left(rpq - \frac{c_1}{K}r - pq^2e_1\right)e_2$$

$$subject\ to\ \begin{cases} q_1e_1 + q_2e_2 \leq r \\ e_1 > 0 \\ e_2\ is\ given \end{cases}$$

The objective is to calculate the fishing effort which maximize the profit of each fisherman for the two problems (1) and (2)

We recall that the point , $e^*(e_1^*, e_2^*)$ is called Nash equilibrium point if and only if $e_1^*$is a solution of problem (1) and $e_2^*$is solution of problem (2) for e_1^*given

The essential conditions of Karush-Kuhn-Tucker applied to problems (1) and (2) require that if $e_1^*$is a solution of problem (1)and if e_2^* is a solution of problem (2) then there exists a constant $m_1, m_2, v \geq$ such that

$$\begin{pmatrix} m_1 \\ m_2 \\ v \end{pmatrix} = \begin{bmatrix} 2\frac{K}{r}pq & \frac{K}{r}pq & -\frac{K}{r} \\ \frac{K}{r}pq & 2\frac{K}{r}pq & -\frac{K}{r} \\ -q & -q & 0 \end{bmatrix}\begin{bmatrix} e_1^* \\ e_2^* \\ 0 \end{bmatrix} + \begin{bmatrix} -Kp + \frac{c_1}{q} \\ -Kp + \frac{c_2}{q} \\ r \end{bmatrix}$$

This problem is equivalent to the linear complementarity problem Foutayni[17] wherethe matrix M is a P matrix

$$M = \begin{bmatrix} 2\frac{K}{r}pq^1 & \frac{K}{r}pq^2 & -\frac{K}{r} \\ \frac{K}{r}pq^1 & 2\frac{K}{r}pq^2 & -\frac{K}{r} \\ -q^1 & -q^2 & 0 \end{bmatrix}$$

and

$$b = \begin{bmatrix} -Kp + \frac{c_1}{q} \\ -Kp + \frac{c_2}{q} \\ r \end{bmatrix}$$

The matrix M is a P-matrix (a matrix with positive principal minor) Uwe Schafer (2004)[18], It is well known that LCP (M p) has a unique solution for every p if and only if M is a P matrix, so LCP (M, b) have one

$$e_1^* = \frac{r}{pK}\frac{c_1}{q^2} \text{ and } \quad e_2^* = \frac{r}{pK}\frac{c_2}{q^2}$$

6. Dissucission and Conclusions:

In this study, the population dynamic of fish with and without harvesting, and with and without predator was considered. The fish population without harvesting is assumed to follow the Logistic Model with constant intrinsic growth rate r, asymptotic growth K and the harvesting rate is constant. As a result, the corresponding harvesting model is also a Logistic model with constant intrinsic growth rate $r - h$ and asymptotic growth $K(r - h)$/r provided that $r > h$. The modelling problem is how to maximize the sustainability of the yield by determining the population growth dynamics to fix the harvesting rate which keeps the population at its maximum growth rate sustainable fish harvesting requires comprehension of the appropriate harvesting models for application. Each of these strategies offers a route to this goal, with its own benefits and limitations. This fishing method, Proportional Harvesting, involves removing a certain proportion of fish from the population each year. It easily adjusts with fluctuations in population size through periods of low populations that may provide a buffer against overfishing and high populations used to increase yield. Consequently, if 40% (H=0.4) is a proportional harvest rate then it would indicate that the population should hover around half the number that can be supported by the environment thereby optimizing growth as well as maintaining sustainability. Its adaptive nature makes it much more resistant to changes in natural variables and uncertainties in population dynamics than any other method. Nevertheless, constant harvesting refers to removing a fixed number of individuals irrespective of population size annually. This helps stabilize annual yields making it easier for organizations to plan economically and for the market to have a steady supply For example an unchanging harvest every year at 9000 fish could keep the population near its target level so long as there are initial.

The scope of studying fish harvesting using constant and proportional models encompasses various ecological, economic, and social dimensions. These models provide frameworks to understand and manage fish populations sustainably. Key areas of application include:

- Ecological Management: Maintaining fish population sizes within sustainable limits to prevent overfishing and ensure the long-term viability of species. Balancing ecosystem health by considering the impacts of harvesting on biodiversity and habitat stability.

- Economic Considerations: Optimizing harvest strategies to maximize economic returns while ensuring sustainability.Assessing the economic implications of different harvesting methods on fishery-dependent communities.
- Policy and Governance: Developing regulations and policies based on scientific models to guide sustainable fishery practices.Implementing adaptive management strategies that incorporate real-time data and feedback mechanisms.

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