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On Product (E, 1) (C, α,β) Summabilityof Fourier Series and its Conjugate Series

By

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Abstract
Summability is defined as a division of mathematical analysis where in an infinite series which is divergent by conventional summation methods is made to converge to a sum say'w' & become summable through dissimilar summation methods. Ernesto Cesaro gave one such method known as C Method in which (C, 1) is the notation for ordinary Cesaro summation & (C, α) is the notation for generalized Cesaro summation. Euler provided summation formula which sums infinite series called (E, 1) summation method. Generalized (E, 1) (C, 1) to (E, 1) (C, α) (α > 0) product summation is given by S.N.Mishra & Harsh Joshi [7]. The objective of this paper is to generalize (E, 1)(C, α) (α > 0) to (E, 1)(C, α , β) (α > 0)(β > -1) so that the series which can't be made summable by (E, 1)(C, 1) & (E, 1)(C, α) methods can be made summable by (E, 1)(C, α , β) (α > 0)(β > -1)methods.

Keywords: (E, 1) means, (C, α) means, (E, 1) (C, α, β) product Means, Fourier series, Conjugate Series.
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1. Introduction

Let $w_0, w_1, w_2, w_3, \ldots .. w_v$ denote partial sums of infinite series Σs_n. The series Σs_n converges to a particular sum 'w' in (C, 1) means if

$$\sigma = \frac{w_0 + w_1 + w_2 + w_3 + \cdots + w_v}{s+1} \to s$$

As s⟶∞. (C, α) Summability of an infinite series Σs_n is given by

$$(C, \alpha) = \frac{1}{\binom{n+\alpha}{\alpha}} \sum_{s=0}^{n} \binom{s+\alpha}{\alpha} w_{n-s} \to w$$

$$(C, \alpha) = \frac{1}{\binom{n+\alpha}{\alpha}} \sum_{s=0}^{n} \binom{n-s+\alpha}{\alpha} w_n \to s \text{ as } n \to \infty.$$

And (C, α, β) summability of an infinite series Σs_n is given by

$$(C, \alpha, \beta) = \frac{1}{\binom{n+\alpha+\beta}{\alpha+\beta}} \sum_{s=0}^{n} \binom{n-s+\alpha+\beta}{\alpha+\beta} w_n \to w \text{ , when } \alpha > 0, \beta > -1.$$

(C, α, β) Summability Method

Let f(x) be any function which is Lebesgue integrable in any finite interval of x ≥ 0 and bounded in some right hand neighborhood of the origin. Let $\partial_{\alpha, \beta\,(x) \text{ and the}}$ (C, α, β) transform of f(x) is defined by

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$$\partial_{\alpha+\beta}(x) = \begin{pmatrix} f(x)\alpha=0 \\ \frac{\Gamma(\alpha+\beta+1)}{\Gamma\alpha\Gamma\beta+1}\frac{1}{x^{\alpha+\beta}}\int_0^x x-y^{\alpha-1}y^{\beta}f(y)dy \qquad (\alpha>0,\beta>-1) \end{pmatrix}$$

If for x >0, the integral defining ∂ α, β (x) exists and if it tends to w, as x tends to infinity, we say that f(x) is summable (C, α, β) to w and we write f(x) →w(C, α, β).

The series Σs_n is Euler summable (E, 1) to a sum w if.

(E; 1) $=\frac{1}{2^s}\sum_{q=0}^{s}\binom{s}{q} w_s \rightarrow w$, as s → ∞.

Then (C, α, β) (E, 1) summability of series Σs_n is given by,

$C_n^{\alpha,\beta}E_v^1 = \frac{1}{\binom{n+\alpha+\beta}{\alpha+\beta}}\sum_{s=0}^{n}\binom{s+\alpha+\beta}{\alpha+\beta}E_v^1 \rightarrow$ w, when α > 0, β>-1 and n → ∞.

Let f(x) be a 2π periodic function of x and Lebesgue integrable in the interval (-π, π). The Fourier series of f(x) is given by

$f(x) \sim \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nx+b_n \sin nx) =\sum_{n=1}^{\infty} A_n(x)$.

The conjugate series is given by

$\sum_{n=1}^{\infty}(a_n \cos nx - b_n \sin nx) = \sum_{n=1}^{\infty} B_n(x)$.

We have following assumptions

$\phi(t) = f(x + t) + f(x - t) - 2 f(x)$. $\Psi(t) = f(x + t) - f(x - t)$

$H_n(t) =\frac{1}{2\pi\binom{n+\alpha+\beta}{\alpha+\beta}} \left| \sum_{s=0}^{n}[\binom{s+\alpha+\beta}{\alpha+\beta}\frac{1}{2^s}\sum_{q=0}^{s}\binom{s}{q}\frac{\sin\left(q+\frac{1}{2}\right)t}{\sin\frac{t}{2}}] \right|$

$\overline{H_n(t)}=\frac{1}{2\pi\binom{n+\alpha+\beta}{\alpha+\beta}} \left| \sum_{s=0}^{n}[\binom{s+\alpha+\beta}{\alpha+\beta}\frac{1}{2^s}\sum_{q=0}^{s}\binom{s}{q}\frac{\cos\left(q+\frac{1}{2}\right)t}{\sin\frac{t}{2}}] \right|$

where $\tau = [\frac{1}{t}]$, where τ denotes greatest integer not greater than $\frac{1}{t}$.

2. Main Theorems

Theorem 1: Let $\{P_n\}$ be a positive monotonic, non increasing sequence of the real constant such that $P_n=\sum_s^n P_s \rightarrow \infty$, n→ ∞. If

$\emptyset(t) = \int_0^t|\emptyset(u)|du = o(\frac{t}{[\alpha(\frac{1}{t})+\beta(\frac{1}{t})]P})$ as $t\rightarrow +0$.

where $\alpha(t)$ and $\beta(t)$ are positive monotonic and non-increasing function of t such that

log (n+1) =O ($[\alpha(n+1)+\beta(n+1)]P_{n+1}$) as n→ ∞.

Then Fourier series is summable to (E, 1) (C, α, β) to f(x).

Theorem 2: Let $\{P_n\}$ be a positive monotonic, non increasing sequence of the real constant such that $P_n=\sum_s^n P_s \rightarrow \infty$, n→ ∞. If

$\varphi(t) = \int_0^t|\varphi(u)|du = o(\frac{t}{[\alpha(\frac{1}{t})+\beta(\frac{1}{t})]P_\tau})$ as n→ ∞

where $\alpha(t)$ and $\beta(t)$are positive monotonic and non-increasing function then conjugate series is summable to $\overline{f(x)}= -\frac{1}{2\pi}\int_0^{2\pi} \varphi(t)\cot(\frac{t}{2})\, dt$.

3. Required Lemmas

Lemma 1: $|H_n(t)| = O(n+1)$. For $0 \le t \le \frac{1}{n+1}$

Proof: For $0 \le t \le \frac{1}{n+1}$, $\sin nt \le n \sin t$

$$H_n(t)\le \frac{1}{2\pi\binom{n+\alpha+\beta}{\alpha+\beta}}\sum_{s=0}^{n}[\binom{s+\alpha+\beta}{\alpha+\beta}\frac{1}{2^s}\sum_{q=0}^{s}\binom{s}{q}\frac{\sin\left(q+\frac{1}{2}\right)t}{\sin\frac{t}{2}}]$$

$$\le \frac{1}{2\pi\binom{n+\alpha+\beta}{\alpha+\beta}}\Big|\sum_{s=0}^{n}[\binom{s+\alpha+\beta}{\alpha+\beta}\frac{1}{2^s}\sum_{q=0}^{s}\binom{s}{q}(2q+1)\Big|$$

$$\le \frac{1}{2\pi\binom{n+\alpha+\beta}{\alpha+\beta}}\Big|\sum_{s=0}^{n}[\binom{s+\alpha+\beta}{\alpha+\beta}\frac{1}{2^s}\sum_{q=0}^{s}\binom{s}{q}(2s+1)\Big|$$

$$\le \frac{(2n+1)}{2\pi\binom{n+\alpha+\beta}{\alpha+\beta}}[\sum_{s=0}^{n}[\binom{s+\alpha+\beta}{\alpha+\beta}]]$$

$$\le \frac{(2n+1)(n+1+\alpha+\beta)}{2\pi(\alpha+\beta+1)}$$

$= O(n+1)$.

Lemma 2: $|H_n(t)| = O(\frac{n+1}{t})$, For $\frac{1}{n+1} \le t \le \pi$.

Proof: For $\frac{1}{n+1} \le t \le \pi$, applying Jordan's lemma $\sin\frac{t}{2} \ge \frac{t}{\pi}$,

$|\sin nt| \le 1$. $$H_n(t)\le \frac{1}{2\pi\binom{n+\alpha+\beta}{\alpha+\beta}}\sum_{s=0}^{n}[\binom{s+\alpha+\beta}{\alpha+\beta}\frac{1}{2^s}\sum_{q=0}^{s}\binom{s}{q}\frac{\sin\left(q+\frac{1}{2}\right)t}{\sin\frac{t}{2}}]$$

$$\le \frac{1}{2t\binom{n+\alpha+\beta}{\alpha+\beta}}\sum_{s=0}^{n}[\binom{s+\alpha+\beta}{\alpha+\beta}\frac{1}{2^s}\sum_{q=0}^{s}\binom{s}{q}]$$

$$\le \frac{1}{2t\binom{n+\alpha+\beta}{\alpha+\beta}}\sum_{s=0}^{n}[\binom{s+\alpha+\beta}{\alpha+\beta}$$

$$\le \frac{1}{2t\binom{n+\alpha+\beta}{\alpha+\beta}}[\sum_{s=0}^{n}[\binom{n+1+\alpha+\beta}{\alpha+1+\beta}]]$$

$$\le \frac{n+\alpha+1+\beta}{2t(\alpha+1+\beta)} \le O(\frac{n+1}{t}).$$

Lemma 3: $|\overline{H_n(t)}| = O(\frac{n+1}{t})$. For $0 \le t \le \frac{1}{n+1}$

Proof: For $0 \le t \le \frac{1}{n+1}\sin\frac{t}{2} \ge \frac{t}{\pi}$ and $|\cos nt| \le 1$,

$$|\overline{H_n(t)}| \le \frac{1}{2\pi\binom{n+\alpha+\beta}{\alpha+\beta}}\sum_{s=0}^{n}[\binom{s+\alpha+\beta}{\alpha+\beta}\frac{1}{2^s}\sum_{q=0}^{s}\binom{s}{q}\frac{\cos\left(q+\frac{1}{2}\right)t}{\sin\frac{t}{2}}]$$

$$\leq \frac{1}{2\pi\binom{n+\alpha+\beta}{\alpha+\beta}}\sum_{s=0}^{n}\left[\binom{s+\alpha+\beta}{\alpha+\beta}\frac{1}{2^s}\sum_{q=0}^{s}\binom{s}{q}\frac{|\cos\left(q+\frac{1}{2}\right)t|}{|\sin\frac{t}{2}|}\right]$$

$$\leq \frac{1}{2\pi\binom{n+\alpha+\beta}{\alpha+\beta}}\sum_{s=0}^{n}\left[\binom{s+\alpha+\beta}{\alpha+\beta}\frac{1}{2^s}\sum_{q=0}^{s}\binom{s}{q}\frac{1}{(\frac{t}{\pi})}\right]$$

$$\leq \frac{1}{2t\binom{n+\alpha+\beta}{\alpha+\beta}}\sum_{s=0}^{n}\left[\binom{s+\alpha+\beta}{\alpha+\beta}\frac{1}{2^s}\sum_{q=0}^{s}\binom{s}{q}\right]$$

$$\leq \frac{1}{2t\binom{n+\alpha+\beta}{\alpha+\beta}}\sum_{s=0}^{n}\left[\binom{s+\alpha+\beta}{\alpha+\beta}\right]$$

$$\leq \frac{n+\alpha+1+\beta}{2t(\alpha+1+\beta)}$$

$$= O\left(\frac{n+1}{t}\right)$$

Lemma 4: For $0 \leq a \leq b \leq \infty$, $0 \leq t \leq \pi$ and for any n, we have $|H_n(t)| = O\left(\frac{n+1}{t}\right)$.

Proof: For $0 \leq \frac{1}{n+1} \leq t \leq \pi$, $\sin\frac{t}{2} \geq \frac{t}{\pi}$,

$$|\overline{H_n(t)}| \leq \frac{1}{2\pi\binom{n+\alpha+\beta}{\alpha+\beta}}\sum_{s=0}^{n}\left[\binom{s+\alpha+\beta}{\alpha+\beta}\frac{1}{2^s}\sum_{q=0}^{s}\binom{s}{q}\frac{\cos\left(q+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right]|$$

$$\leq \frac{1}{2\pi\binom{n+\alpha+\beta}{\alpha+\beta}}\sum_{s=0}^{n}\left[\binom{s+\alpha+\beta}{\alpha+\beta}\frac{1}{2^s}\sum_{q=0}^{s}\binom{s}{q}\frac{|\cos\left(q+\frac{1}{2}\right)t|}{|\sin\frac{t}{2}|}\right]$$

$$\leq \frac{1}{2\pi\binom{n+\alpha+\beta}{\alpha+\beta}}\sum_{s=0}^{n}\left[\binom{s+\alpha+\beta}{\alpha+\beta}\frac{1}{2^s}\sum_{q=0}^{s}\binom{s}{q}\frac{\cos\left(q+\frac{1}{2}\right)t}{\frac{t}{\pi}}\right]$$

$$\leq \frac{1}{2t\binom{n+\alpha+\beta}{\alpha+\beta}}\sum_{s=0}^{n}\left[\binom{s+\alpha+\beta}{\alpha+\beta}\frac{1}{2^s}\sum_{q=0}^{s}\binom{s}{q}\operatorname{Re}\exp i\left(q+\frac{1}{2}\right)\right]$$

$$\leq \frac{1}{2t\binom{n+\alpha+\beta}{\alpha+\beta}}\sum_{s=0}^{n}\left[\binom{s+\alpha+\beta}{\alpha+\beta}\frac{1}{2^s}\sum_{q=0}^{s}\binom{s}{q}\operatorname{Re}\left(\exp(iqt)\right)\right]\left[|\exp(i\tfrac{t}{2})|\right]$$

$$\leq \frac{1}{2t\binom{n+\alpha+\beta}{\alpha+\beta}}\sum_{s=0}^{n}\left[\binom{s+\alpha+\beta}{\alpha+\beta}\frac{1}{2^s}\operatorname{Re}\sum_{q=0}^{s}\binom{s}{q}(\exp(iqt))\right]$$

$$\leq \frac{1}{2t\binom{n+\alpha+\beta}{\alpha+\beta}}\sum_{s=0}^{\tau-1}\left[\binom{s+\alpha+\beta}{\alpha+\beta}\frac{1}{2^s}\operatorname{Re}\sum_{q=0}^{s}\binom{s}{q}(\exp(iqt))\right]$$

$$+\ \frac{1}{2t\binom{n+\alpha+\beta}{\alpha+\beta}}\sum_{\tau-1}^{n}\left[\binom{s+\alpha+\beta}{\alpha+\beta}\frac{1}{2^s}\operatorname{Re}\sum_{q=0}^{s}\binom{s}{q}(\exp(iqt))\right]|$$

$$= \overline{H}_1 + \overline{H}_2$$

$$\overline{H}_1 \leq \frac{1}{2t\binom{n+\alpha+\beta}{\alpha+\beta}}\left|\sum_{s=0}^{\tau-1}\left[\binom{s+\alpha+\beta}{\alpha+\beta}\frac{1}{2^s}\operatorname{Re}\left(\sum_{q=0}^{s}\binom{s}{q}\right)(\exp(iqt))\right]\right|$$

$$\overline{H}_1 \leq \frac{1}{2t\binom{n+\alpha+\beta}{\alpha+\beta}}\left|\sum_{s=0}^{\tau-1}\left[\binom{s+\alpha+\beta}{\alpha+\beta}\frac{1}{2^s}\operatorname{Re}\left(\sum_{q=0}^{s}\binom{s}{q}\right)\right]\right|\,||(\exp(iqt))||$$

$\overline{H}_1 \leq \frac{1}{2t\binom{n+\alpha+\beta}{\alpha+\beta}} \sum_{s=0}^{\tau-1} [\binom{s+\alpha+\beta}{\alpha+\beta}]$

$\overline{H}_1 \leq \frac{n+1}{2t}$. Similarly, $\overline{H}_2 \leq \frac{n+1}{2t}$.

4. Proof of Theorems

Proof of Theorem 1: Using Riemann Lebesgue Theorem and Titchmarsh[4]. $S_n(f; x)$ of the series is given by

$S_n(f; x) - f(x) = \frac{1}{2\pi} \int_0^{\pi} \emptyset(t) \frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}} dt$

Then (C,α, β) summation of $S_n(f; x)$ is given by

$C_n^{\alpha,\beta}$- f(x) = $\frac{1}{2\pi\binom{n+\alpha+\beta}{\alpha+\beta}} \int_0^{\pi} \emptyset(t) [\sum_{s=0}^{n} [\binom{s+\alpha+\beta}{\alpha+\beta} \frac{\sin(s+\frac{1}{2})t}{\sin\frac{t}{2}} dt]$

Now (E, 1) (C,α, β) summation of $S_n(f; x)$ is given by

$E_v^1 C_n^{\alpha,\beta}$- f(x) = $\frac{1}{2\pi\binom{n+\alpha+\beta}{\alpha+\beta}} \int_0^{\pi} \emptyset(t) \sum_{s=0}^{n} [\binom{s+\alpha+\beta}{\alpha+\beta} \frac{1}{2^s} \sum_{s=0}^{n} \binom{s}{q} \frac{\sin(s+\frac{1}{2})t}{\sin\frac{t}{2}} dt]$

$= \int_0^{\pi} \emptyset(t) H_n(t) dt.$

Now, we have to prove that,

$= \int_0^{\pi} \emptyset(t) H_n(t) dt = o(1)$ as $n \to \infty$.

For $0 < \delta < \pi$,

$= \int_0^{\frac{1}{n+1}} \emptyset(t) H_n(t) dt + \int_{\frac{1}{n+1}}^{\delta} \emptyset(t) H_n(t) dt + \int_{\delta}^{\pi} \emptyset(t) H_n(t) dt$

$= I_1 + I_2 + I_3.$

Then, $I_1 \leq \int_0^{\frac{1}{n+1}} |\emptyset(t)| |H_n(t)| dt.$

From Lemma 1, $| H_n(t) | = O(n+1)$

$= O(n+1) [\int_0^{\frac{1}{n+1}} |\emptyset(t)| |H_n(t)| dt]$

Also, $\emptyset(t) = o(\frac{1}{\frac{1}{t}(\alpha(\frac{1}{t})+\beta(\frac{1}{t}))P_\tau})$

$= O(n+1) [o(\frac{1}{\frac{1}{t}\left(\alpha(\frac{1}{t})+\beta(\frac{1}{t})\right)P_\tau})_0^{\frac{1}{n+1}}]$

$= O(n+1)\, o(\frac{1}{n[\alpha(n+1)+\beta(n+1)]P_{n+1}})$

$= o(\frac{1}{[\alpha(n+1)+\beta(n+1)]P_{n+1}})$

$= o(\frac{1}{\log(n+1)})$

$= o\,(1) \qquad \text{as } n \to \infty$

From Lemma 2,

$H_n(t) = o\,(\frac{n+1}{t})$.

Integrating by parts,

$= O\,(n+1)\, O\,[\,(\frac{\emptyset(t)}{t})_{\frac{1}{n+1}}^{\delta} + \int_{\frac{1}{n+1}}^{\delta} \frac{\emptyset(t)}{t^2} dt]$.

By Lemma 1, $\emptyset(t) = o(\frac{1}{\frac{1}{t}\left(\alpha\left(\frac{1}{t}\right)+\beta\left(\frac{1}{t}\right)\right)P_\tau})$

Let $t = \frac{1}{u} \to dt = -\frac{dt}{u^2}$.

Now we have,

$=O(n+1)O[o(\frac{1}{[\alpha(n)+\beta(n)]P_n}) + \int_{\delta}^{\frac{1}{n+1}} o(\frac{1}{u[\alpha(u)+\beta(u)]P_u})du$

$= O\,(n+1)\; o\,(\frac{1}{[\alpha(n+1)+\beta(n+1)]P_{n+1}}) + o\,(\frac{1}{[\alpha(n+1)+\beta(n+1)]P_{n+1}})$

$=O\,(n+1)\,[o\,(\frac{1}{\log(n+1)}) + o\,(\frac{1}{\log(n+1)})]$

$= o\,(\frac{n+1}{\log(n+1)}) + o\,(\frac{n+1}{\log(n+1)})$

$= o\,(1) + o\,(1) \qquad \text{as } n\to \infty$.

Similarly, $I_3 = \int_0^{\pi} \emptyset(t) H_n(t)\, dt = o\,(1) \quad \text{as } n\to \infty$.

Proof of Theorem 2: If $\overline{S_n(f;x)}$ denotes partial sum of conjugate series then from Riemann Lebesgue theorem and using Titchmarsh [4].

$\overline{S_n(f;x)} = \frac{1}{2\pi} \int_0^{\pi} \emptyset(t)\, \frac{\cos(n+\frac{1}{2})t}{\sin\frac{t}{2}} dt$

Then, $\overline{C_n^{\alpha,\beta}}$ summation of $\overline{S_n(f;x)}$ is given by,

$\overline{C_n^{\alpha,\beta}} - f(x) = \frac{1}{2\pi\binom{n+\alpha+\beta}{\alpha+\beta}} \int_0^{\pi} \varphi(t) [\sum_{s=0}^{n} [\binom{s+\alpha+\beta}{\alpha+\beta} \frac{\cos(s+\frac{1}{2})t}{\sin\frac{t}{2}} dt]$

Now (E, 1) (C,α, β) summation of $\overline{S_n(f;x)}$ is given by

$\overline{E_v^1 C_n^{\alpha,\beta}}$

$\overline{E_v^1 C_n^{\alpha,\beta}} - f(x)$

$= \frac{1}{2\pi\binom{n+\alpha+\beta}{\alpha+\beta}} \int_0^{\pi} \varphi(t) \sum_{s=0}^{n} [\binom{s+\alpha+\beta}{\alpha+\beta} \frac{1}{2^s} \sum_{s=0}^{n} \quad \binom{s}{q} \frac{\cos(s+\frac{1}{2})t}{\sin\frac{t}{2}} dt]$

$= \int_0^{\pi} \varphi(t)[\overline{H_n(t)}]dt.$

Now, we have to prove that,

$= \int_0^{\pi} \varphi(t)[\overline{H_n(t)}]dt$

$$= o(1) \qquad \text{as } n \to \infty.$$

For $0 < \delta < \pi$,

$= \int_0^{\pi} \varphi(t)[\overline{H_n(t)}]dt$

$= \int_0^{\frac{1}{n+1}} \varphi(t)[\overline{H_n(t)}]dt + \int_{\frac{1}{n+1}}^{\delta} \varphi(t)[\overline{H_n(t)}]dt + \int_{\delta}^{\pi} \varphi(t)[\overline{H_n(t)}]dt$

$= I_1' + I_2' + I_3'.$

Where $I_1' = \int_0^{\frac{1}{n+1}} \varphi(t)[\overline{H_n(t)}]dt$

$I_2' = \int_{\frac{1}{n+1}}^{\delta} \varphi(t)[\overline{H_n(t)}]dt$

and $I_3' = \int_{\delta}^{\pi} \varphi(t)[\overline{H_n(t)}]dt$

then, $I_1 \le \int_0^{\frac{1}{n+1}} |\varphi(t)||H_n(t)|dt.$

$\le O\left(\frac{n+1}{t}\right)\left[\int_0^{\frac{1}{n+1}} |\varphi(t)|\, dt\right]$

$\le O(n)\int_0^{\frac{1}{n+1}} |\varphi(t)|\, dt$

Also, By Theorem 2,

$\varphi(t) = o\left(\frac{1}{\frac{1}{t}(\alpha\left(\frac{1}{t}\right)+\beta\left(\frac{1}{t}\right))P_\tau}\right)$

$= O(n)\int_0^{\frac{1}{n}} o\left(\frac{1}{\frac{1}{t}\left(\alpha\left(\frac{1}{t}\right)+\beta\left(\frac{1}{t}\right)\right)P_\tau}\right) dt$

$= O(n+1)\, o\left(\frac{1}{n[\alpha(n+1)+\beta(n+1)]P_{n+1}}\right)$

$= O(n+1)\, o\left(\frac{1}{[\alpha(n+1)+\beta(n+1)]P_{n+1}}\right)$

As $\log n = o\left[(\alpha(n+1) + \beta(n+1))P_{n+1}\right]$

$= o\left(\frac{n+1}{\log(n+1)}\right)$

$= o(1) \qquad \text{as } n \to \infty$

Now we consider,

$I_2' = \int_{\frac{1}{n+1}}^{\delta} |\varphi(t)||\overline{H_n(t)}|$

From Lemma 4, $\overline{H_n(t)} = o\left(\frac{n+1}{t}\right).$

Integrating by parts,

$= O\ (n+1)\ O[\int_{\frac{1}{n+1}}^{\delta} \frac{\varphi(t)}{t} dt]$

$= O\ (n+1)\ O\ [\ (\frac{\varphi(t)}{t})_{\frac{1}{n+1}}^{\delta} + \int_{\frac{1}{n+1}}^{\delta} \frac{\varphi(t)}{t^2} dt].$

By Theorem 2,

$\varphi(t) = o(\frac{1}{\frac{1}{t}\left(\alpha\left(\frac{1}{t}\right)+\beta\left(\frac{1}{t}\right)\right)P_\tau})$

$=O\ (n+1)\ O\ [o(\frac{1}{\left(\alpha\left(\frac{1}{t}\right)+\beta\left(\frac{1}{t}\right)\right)P_\tau})_{\frac{1}{n+1}}^{\delta} + \int_{\frac{1}{n+1}}^{\delta} o(\frac{1}{t\left(\alpha\left(\frac{1}{t}\right)+\beta\left(\frac{1}{t}\right)\right)P_\tau}) dt]$

$= O\ (n+1)\ o\ [(\frac{1}{[\alpha(n+1)+\beta(n+1)]P_{n+1}}) + \int_{\frac{1}{n+1}}^{\delta} o(\frac{1}{t\left(\alpha\left(\frac{1}{t}\right)+\beta\left(\frac{1}{t}\right)\right)P_\tau}) dt]$

Let $t = \frac{1}{u} \rightarrow dt = -\frac{dt}{u^2}$

$=O\ (n+1)\ O\ [o\ (\frac{1}{[\alpha(n+1)+\beta(n+1)]P_{n+1}}) + \int_{\delta}^{\frac{1}{n+1}} o(\frac{1}{u[\alpha(u)+\beta(u)]P_u}) du]$

$= O(n+1)O[o(\frac{1}{[\alpha(n+1)+\beta(n+1)]P_{n+1}}) + o(\frac{1}{n[\alpha(n+1)+\beta(n+1)]P_{n+1}}) \int_{\delta}^{n+1} du]$

$=O\ (n+1)\ O\ [o\ (\frac{1}{[\alpha(n+1)+\beta(n+1)]P_{n+1}}) + o(\frac{1}{[\alpha(n+1)+\beta(n+1)]P_{n+1}})]$

By Theorem 1,

$\text{Log}\ (n+1) = o\ (\alpha(n+1)P_{n+1})$

$= O\ [o\ (\frac{n+1}{\log(n+1)}) + o\ (\frac{n+1}{\log(n+1)})]$

$= o\ (\frac{n+1}{\log(n+1)}) + o\ (\frac{n+1}{\log(n+1)})$

$= o\ (1) + o\ (1)$ as $n \rightarrow \infty$.

$= o\ (1)$ as $n \rightarrow \infty$.

By Regularity Condition of Summability,

$I_3' \leq \int_0^{\pi} |\varphi(t)||\overline{H_n(t)}| dt = o\ (1)$, as $n \rightarrow \infty$.

Then, $\overline{E_v^1 C_n^{\alpha,\beta}} - \overline{f(x)} = o\ (1)$. as $n \rightarrow \infty$, completes proof of theorem.

The above result generalizes to (E, 1) (C,α) when $\beta = 0$ [7].

5. Declaration

The authors declare that there is no conflict of interest regarding the publication of this paper.

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