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A Solution of One-dimensional Advection Dispersion Equation for Concentration of Water Pollutants with Non-zero Dispersion Coefficient

By

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Abstract

In this paper, one-dimensional advection- dispersion equation for concentration of the water pollutant in a small river has been presented by taking none-zero dispersion coefficient. To solve the advection-dispersion equation in unsteady state condition, the Laplace transform technique has been used. It has been found that the concentration of the water pollutant in a small river increases exponentially as the time or the downwind distance or velocity of water pollutant increases.

Keywords: Water pollutant, concentration, Laplace transform, unsteady state.

Introduction

Water is essential for all forms of life. It is a special gift to us from nature. For agriculture, live stock production, forestry, industrial products, fisheries etc, water is essential. The deterioration of water quality is a growing concern today due to significant environmental changes, industrialization and increasing human activity. Water quality is affected by various factors such as: urbanization, sewage and other oxygen demanding waste, industrial wastes, agro-chemical wastes, nutrient enrichment, thermal pollution, oil spillage, the disruption of sediments, acid rain pollution and radioactive waste [1,2].

Human activities as well as natural activities both have the potential to contaminate water. The issue of human health in different parts of the world is closely associated with the environmental and water pollution. The health implications of water pollutants are numerous and varied. Exposure to contaminated water can result in acute and chronic health issues depending on the type of contaminants, its concentration as well as duration of exposure. Water borne diseases can cause dehydration, gastro-intestinal disorders and even life-threatening condition. Long term exposure of certain contaminants can lead to organ damage, reproductive issues and increased cancer risks. Various kinds of studies are available in literature related to water pollutants. Many investigators have studied the concentration profile of water pollutants by describing how pollutants are transported by water flow (advection) and spread due to concentration gradient [3-10]. Mathematical studies on water pollutant concentration involve developing and analyzing models to predict how pollutants spread and accumulate in water

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bodies. These models often use advection-dispersion equation, sometimes coupled with reaction terms to represent pollutant decay and interactions with dissolved oxygen. Numerical solutions to these equations, obtained through simulation, help researchers in understanding pollutant behavior and getting strategies for water quality management. These equations are often solved numerically by using the finite difference method or Runge-Kutta method to obtain approximate solutions for pollutant concentration over the time and space. Simulated results are compared with real-world data (if available) to validate the model and assess its accuracy.

Aral and Liao [1] have given analytical solution for two-dimensional transport equation with dependent dispersion coefficients. Kumar et al. [3,4] have given analytical solution of one-dimensional advection-diffusion equation with variable coefficients in a finite domain. Mourad et al. [5] have studied the effect of added pollutant along a river on pollutant concentration by taking one-dimensional advection-diffusion equation. Pimpumchat et al. [7] have presented a mathematical model for pollution in a river and its remediation. Savovic and Djordjevich [8,9] respectively have given the numerical simulation of one-dimensional advection-diffusion equation and dispersion of pulse type input concentration in semi-infinite media.

Wadi et al. [9] have given analytical solutions for one-dimensional advection-diffusion equation for the pollutant concentration.

In view of the above, the main focus of this paper is to solve the advection-dispersion equation in a finite domain by using the Laplace transform technique and study the behavior of concentration of water pollutant in downwind distance with non-zero dispersion coefficient along a small river.

Mathematical Formulation and Solution

The concentration of water pollutant in unsteady flow of water can be described by a partial differential equation given below (Pimpunchat et al. [6,7]):

$$\frac{\partial(AC)}{\partial t} = D_x \frac{\partial^2(AC)}{\partial x^2} - \frac{\partial(UAC)}{\partial x} - k_1 \frac{X}{X+k} AC + q; \quad 0 \le x < L, \quad t > 0 \tag{1}$$

where U is the water velocity in x-direction, $C(x,t)$ is the concentration of water pollutant, D_x is the dispersion coefficient of pollutant in x-direction which is non-zero, q is the added pollutant rate along the river, k_1 is the degradation rate coefficient of pollutant, k is the half saturated oxygen demand concentration for pollutant decay, A is the cross-section of area of river and X is the concentration of the dissolved oxygen within the river.

We consider a small river in homogeneous system and assume the parameters $U,\ k_1,A$ as constants over time and space and we take $k = 0,\ q = 0$.

Applying above conditions, equation (1) becomes:

$$\frac{\partial(C)}{\partial t} = D_x \frac{\partial^2(C)}{\partial x^2} - \frac{\partial(UC)}{\partial x} - k_1 C; \quad 0 \le x \le L, \quad t > 0 \tag{2}$$

Now, equation (2) is solved under the following conditions:

$$C(x,t) = 0 \;; \qquad x \ge 0,\; t = 0, \tag{3}$$

$$C(x,t) = p \;; \qquad x = 0 \quad and \quad \frac{\partial C(x,t)}{\partial x} = 0\,,\; x = L \tag{4}$$

where p is the velocity of water pollutant at the origin.

We apply Laplace transform technique to solve equation (2) subject to initial condition (3) and boundary condition (4).

Now, taking the Laplace transform of equations (2) and (4), we have

$$D_x \frac{\partial^2 \overline{C}}{\partial x^2} - U \frac{\partial \overline{C}}{\partial x} + C(x,0) - (k_1 + s)\overline{C} = 0; \quad 0 \le x \le L, \quad p > 0 \tag{5}$$

and $$\overline{C}(0,s) = \frac{p}{s} \;,\; \frac{\partial \overline{C}(0,s)}{\partial x} = 0 \tag{6}$$

where s is the Laplace transform variable, and bar(-) denotes the Laplace transform of the corresponding function.

By using (3), equation (5) can be re-written as follows:

$$D_x \frac{\partial^2 \overline{C}}{\partial x^2} - U \frac{\partial \overline{C}}{\partial x} - (k_1 + s)\overline{C} = 0$$

or $$[D_x \frac{\partial^2}{\partial x^2} - U \frac{\partial}{\partial x} - (k_1 + s)]\overline{C} = 0 \tag{7}$$

The solution of equation (7) is given by

$$\overline{C}(x,s) = C_1 e^{\left(\delta + \sqrt{\delta + \frac{k_1+s}{D_x}}\right)x} + C_2 e^{\left(\delta - \sqrt{\delta + \frac{k_1+s}{D_x}}\right)x} \;; \tag{8}$$

where $\delta = \frac{U}{2D_x}$ and C_1 and C_2 are some constants.

Now, applying the condition (6) to equation (8) and simplifying, we get

$$c_1 = \frac{p}{2s}[\frac{\delta}{\sqrt{\delta + \frac{k_1+s}{D_x}}} + 1] \quad \text{and} \quad c_2 = \frac{p}{s} - \frac{p}{2s}[\frac{\delta}{\sqrt{\delta + \frac{k_1+s}{D_x}}} + 1]$$

Using this value of c_1 and c_2 in equation (8), we get

$$\overline{C}(x,s) = \frac{p}{2s}\left[\frac{\delta}{\sqrt{\delta+\frac{k_1+s}{D_x}}}+1\right]e^{\left(\delta+\sqrt{\delta+\frac{k_1+s}{D_x}}\right)x} + \left[\frac{p}{s}-\frac{p}{2s}\left\{\frac{\delta}{\sqrt{\delta+\frac{k_1+s}{D_x}}}+1\right\}\right]e^{\left(\delta-\sqrt{\delta+\frac{k_1+s}{D_x}}\right)x}$$

On simplifying, we have

$$\overline{C}(x,s) = \frac{p}{2s}\frac{\delta}{\sqrt{\delta+\frac{k_1+s}{D_x}}}e^{\left(\delta+\sqrt{\delta+\frac{k_1+s}{D_x}}\right)x} + \frac{p}{2s}e^{\left(\delta+\sqrt{\delta+\frac{k_1+s}{D_x}}\right)x} + \frac{p}{s}e^{\left(\delta-\sqrt{\delta+\frac{k_1+s}{D_x}}\right)x} - \frac{p}{2s}\frac{\delta}{\sqrt{\delta+\frac{k_1+s}{D_x}}}e^{\left(\delta-\sqrt{\delta+\frac{k_1+s}{D_x}}\right)x} - \frac{p}{2s}e^{\left(\delta-\sqrt{\delta+\frac{k_1+s}{D_x}}\right)x} \quad (9)$$

Taking inverse Laplace transform of equation (9), we have

$$C(x,t) = \frac{p}{2}\delta e^{\delta x}\sqrt{D_x}\frac{1}{\sqrt{\pi x}}e^{-(\delta^2 D_x+k_1)x}x\frac{e^{\left[(\delta^2 D_x+k_1)t+\frac{x^2}{4D_x t}\right]}}{2\sqrt{\pi D_x t^3}} + \frac{p}{2}e^{\delta x}e^{-(\delta^2 D_x+k_1)x}x\frac{e^{[(\delta^2 D_x+k_1)t+\frac{x^2}{4D_x t}]}}{2\sqrt{\pi D_x t^3}} + pe^{\delta x}x\frac{e^{[-(\delta^2 D_x+k_1)t-\frac{x^2}{4D_x t}]}}{2\sqrt{\pi D_x t^3}} - \frac{p}{2}\delta e^{\delta x}\sqrt{D_x}\frac{1}{\sqrt{\pi x}}e^{-(\delta^2 D_x+k_1)x}x\frac{e^{[-(\delta^2 D_x+k_1)t-\frac{x^2}{4D_x t}]}}{2\sqrt{\pi D_x t^3}} - \frac{p}{2}e^{\delta x}\sqrt{D_x}x\frac{e^{[-(\delta^2 D_x+k_1)t-\frac{x^2}{4D_x t}]}}{2\sqrt{\pi D_x t^3}} \quad (10)$$

On simplifying, we have

$$C(x,t) = \frac{p}{2}x\delta e^{\delta x}\frac{1}{4\pi t^{3/2}}e^{-(\delta^2 D_x+k_1)}x.\left[e^{[(\delta^2 D_x+k_1)t+\frac{x^2}{4D_x t}]} - e^{-[(\delta^2 D_x+k_1)t+\frac{x^2}{4D_x t}]}\right] + \frac{p}{4\sqrt{\pi D_x t^3}}e^{\delta x}x\left[e^{\left[(\delta^2 D_x+k_1)t+\frac{x^2}{4D_x t}\right]} + e^{-\left[(\delta^2 D_x+k_1)t+\frac{x^2}{4D_x t}\right]}\right] - \frac{p}{4\sqrt{\pi t^3}}e^{\delta x}x\left[e^{-\left[(\delta^2 D_x+k_1)t+\frac{x^2}{4D_x t}\right]}\right] \quad (11)$$

Results and Discussion

The concentration $C(x,t)$ in kg/m^3 obtained from equation (11) is graphically shown under some parametric values used in the equation. We use the time in days and take the parametric values $U = 1, D_x = 1, k_1 = 2$. We also set the value of p *as* $0.0001,\ 0.0002$ and $0.0003\ kg/m^3$.

Figure 1 represents the concentration profile against the distance ($0 \leq x \leq 1$) for different values of time (t) and constant velocity (p) of water pollutant at the origin. It is seen that as the t increases, the concentration $C(x, t)$ of water pollutant increases exponentially. The effect of time is very small near the source and dominant near the downstream.

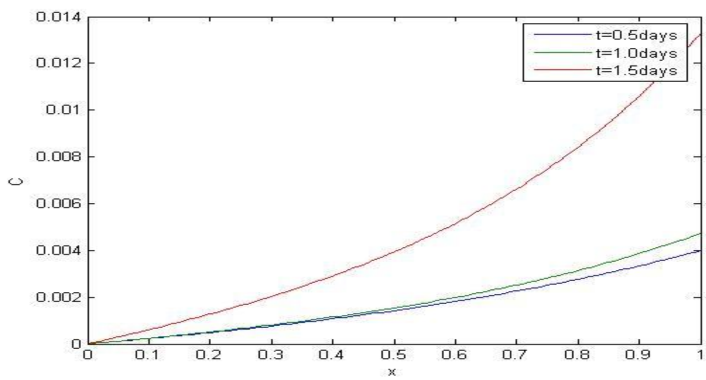

Fig.1. Variation of concentration for constant p with different t.

Figure 2 represents the concentration profile against the distance ($0 \leq x \leq 2$) for constant value of t and different values of p. As the downwind distance x increases, the concentration of water pollutant increases exponentially. Again, as the velocity of water pollutant at the origin increases, the concentration of water pollutant increases at any cross section of the area of the river.

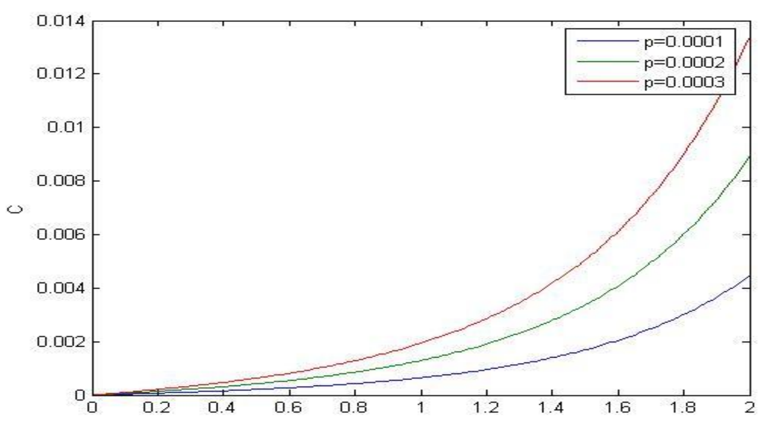

Fig.2. Variation of concentration for different p and constant t.

Conclusion

An analytical solution for the unsteady one-dimensional advection-dispersion equation describing the concentration of water pollutants has been studied by using Laplace transform technique. For the model, we have assumed non- zero dispersion coefficient along the small river and added pollutants zero. It has been observed that the concentration of water pollutant increases exponentially with increasing time. It has also been observed that the concentration of water pollutant increases exponentially as the downwind distance or the velocity of water pollutant along the river increases.

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