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Convergence of Picard Iteration to a Fixed Point in Soft Metric Space

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Abstract

The aim of this paper is to introduce Picard Iteration in soft metric space when the set of parameters is a finite set. We define the Picard operator and show its strong and weak convergence in soft metric space. This paper includes soft metric extensions of several important theorems of Picard iteration for metric space. Throughout the paper a comprehensive set of examples illustrating the discussed topics are presented.

Keywords: Soft mapping, Soft metric space, Soft contraction, Picard iteration, Fixed point, Picard operator.

Mathematics Subject Classification: 47H10, 47J25

1. Introduction

Zadeh [25] initiated fuzzy set theory which is an important tool to solve uncertainties and ambiguities. The contribution made by probability theory, Fuzzy set theory, vague sets, rough sets and interval mathematics to deal with uncertainties is of vital importance, but these theories have their own limitations. To overcome these peculiarities, in 1999, Molodtsov initiated the soft set theory. It is a branch of mathematics in which we have a mathematical tool to deal with real-world uncertainties and problems. It provides sufficient tools to solve the uncertainties in data and to represent it in a useful way. [4], [5], [7], [17], [19] are some papers on soft sets and [15], [20], [23] are some papers in which combination of soft sets with fuzzy sets have been discussed. Most of the physical problems of applied science and engineering are usually formulated in the form of fixed-point equations. Hence, in the present paper, we want to investigate a fixed point in soft set theory for a certain class of newly defined soft mappings. Soft set theory has attracted several mathematicians, economists and computer scientists ([3], [4], [19], etc.). Soft set theory has applications in decision-making, demand analysis, forecasting, information sciences, mathematics and other disciplines ([11], 12], [13], [16], [22], [26]). In [9] Das and Samanta introduced the notion of soft real sets and soft real numbers and discussed their properties. Also, Das and Samanta [10] introduced the concept of soft metrics. Soft contraction mapping based on soft elements of soft metric spaces is introduced by Abbas et al. [1]. In [8] Chen and Lin proved Meir- Keeler fixed point theorem in a soft metric space. In [6], Let X be any set and $T: X \to X$ a self-map. For any given $x \in X$, we define $T^n x$ inductively by $T^0(x) = x$ and $T^{n+1}(x) = T(T^n(x))$. We call $T^n(x)$ the n^{th} iterate of x under T. The mapping $T^n (n \geq 1)$ is called the n^{th} iterate of T. For any $x \in X$, the sequence $\{x_n\} \subset X$ given by

$$x_n = Tx_{n-1} = T^n x_0; \qquad n = 1, 2, 3, 4, 5, \dots \dots$$

is called a sequence of successive approximations with an initial value x_0. It is also known as the Picard iteration, starting at x_0. The study of an iterative process to appropriate the solution of a fixed point is an

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active area of research (see eg. [25], [21], [3], [12], [22] and the references therein). The Picard iteration scheme is one of the simplest iteration schemes used to approximate the solution of fixed point equations involving a non-linear contractive operator. In 2016, Abbas *et al*., show that under some restriction, each soft metric induces a usual metric and deduces in a direct way soft metric versions of several important fixed point theorems for metric spaces. In this paper, we deduce in a direct way soft metric version of Picard iteration for (complete) metric spaces. Also, we discuss its strong and weak convergence in soft metric space. The paper includes many examples illustrating present concepts and showing the necessity of some assumptions.

2. Preliminaries

Definition 2.1: A soft real set denoted by $(\hat{f}, A)$, or simply by $\hat{f}$, is a mapping $\hat{f}: A \to B(\mathbb{R})$, where $B(\mathbb{R})$ is the nonempty bounded subsets of $\mathbb{R}$. If $\hat{f}$ is a single-valued mapping on $A \subset E$ taking values in $\mathbb{R}$, then the pair $(\hat{f}, A)$ or simply $\hat{f}$, is called a soft element of $\mathbb{R}$ or a soft real number. If $\hat{f}$ is a single-valued mapping on $A \subset E$ taking values in the set $\mathbb{R}^+$ of nonnegative real numbers, then the pair $(\hat{f}, A)$, or simply $\hat{f}$, is called a nonnegative soft real number. We shall denote the set of nonnegative soft real numbers (corresponding to A) by $\mathbb{R}(A)^*$. A constant soft real number $\overline{c}$ is a soft real number such that for each $e \in A$, we have $\overline{c}(e) = c$, where c is some real number.

Definition 2.2: For two soft real numbers $\hat{f}$, $\hat{g}$, we say that

(1) $\hat{f} \,\tilde{\leqq}\, \hat{g}$ if $\hat{f}(e) \leq \hat{g}(e)$, for all $e \in A$,

(2) $\hat{f} \,\tilde{\geqq}\, \hat{g}$ if $\hat{f}(e) \geq \hat{g}(e)$, for all $e \in A$

(3) $\hat{f} \,\tilde{<}\, \hat{g}$ if $\hat{f}(e) < \hat{g}(e)$, for all $e \in A$, and

(4) $\hat{f} \,\tilde{>}\, \hat{g}$ if $\hat{f}(e) > \hat{g}(e)$, for all $e \in A$.

Remark: Recall that if f is a soft mapping from a soft set (F, A) to a soft set (G, B) (denoted by $f: (F, A) \tilde{\to} (G, B)$), then for each soft point $F_\lambda^x \,\tilde{\in}\, (F, A)$ there exists only one soft point $G_\mu^y \,\tilde{\in}\, (G, B)$ such that $f(F_\lambda^x) = G_\mu^y$.

The definition of a soft metric introduced in [10] is given below.

Definition 2.3: Let $\mathcal{U}$ be a universe, A be a nonempty subset of parameters and $\mathcal{U}$ be the absolute soft set, i.e., $F(\lambda) = \mathcal{U}$ for all $\lambda \in A$, where $(F, A) = \tilde{\mathcal{U}}$. A mapping $d: SP(\tilde{\mathcal{U}}) \times SP(\tilde{\mathcal{U}}) \to \mathbb{R}(A)^*$ is said to be a soft metric on $\tilde{\mathcal{U}}$ if for any $U_\lambda^x, U_\mu^y, U_\gamma^z \,\tilde{\in}\, SP(\tilde{\mathcal{U}})$ (equivalently, $U_\lambda^x, U_\mu^y, U_\gamma^z \,\tilde{\in}\, \tilde{\mathcal{U}}$), the following hold:

1. $d(\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y) \,\tilde{\geqq}\, \overline{0}$.
2. $d(\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y) = \overline{0}$ if and only if $\mathcal{U}_\lambda^x = \mathcal{U}_\mu^y$.
3. $d(\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y) = d(\mathcal{U}_\mu^x, \mathcal{U}_\lambda^y)$.
4. $d(\mathcal{U}_\lambda^x, \mathcal{U}_\gamma^z) \,\tilde{\leqq}\, d(\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y) + d(\mathcal{U}_\mu^y, \mathcal{U}_\gamma^z)$.

The soft set $\tilde{\mathcal{U}}$ endowed with a soft metric d is called a soft metric space and is denoted by $(\tilde{\mathcal{U}}, d, A)$, or simply by $(\tilde{\mathcal{U}}, d)$ if no confusion arises.

See [9] for several basic properties of the structure of soft metric spaces. In order to help the reader, we recall the following notions, which will be used later on.

Given a soft metric space $(\tilde{\mathcal{U}}, d)$, a net $\left\{\mathcal{U}_{\lambda_\alpha}^{x_\alpha}\right\}_{\alpha\in\Lambda}$ of soft points in $\tilde{\mathcal{U}}$ will be simply denoted by $\left\{\mathcal{U}_{\lambda,\alpha}^{x}\right\}_{\alpha\in\Lambda}$. In particular, a sequence $\left\{\mathcal{U}_{\lambda_n}^{x_n}\right\}_{n\in\mathbb{N}}$ of soft points in $\tilde{\mathcal{U}}$ will be denoted by $\left\{\mathcal{U}_{\lambda,n}^{x}\right\}_{n}$.

Definition 2.4: Let $(\tilde{\mathcal{U}}, d)$ be a soft metric space. A sequence $\{\mathcal{U}_{\lambda,n}^x\}_n$ of soft points in $\tilde{\mathcal{U}}$ is said to be convergent in $(\tilde{\mathcal{U}}, d)$ if there is a soft point $U_\mu^y \tilde{\in} \tilde{\mathcal{U}}$ such that $d(\mathcal{U}_{\lambda,n}^x, \mathcal{U}_\mu^y) \to \overline{0}$ as $n \to \infty$. This means that for every $\hat{\epsilon} \tilde{>} \overline{0}$, chosen arbitrarily, there exists an $m \in \mathbb{N}$ such that $d(\mathcal{U}_{\lambda,n}^x, \mathcal{U}_\mu^y) \tilde{<} \hat{\epsilon}$, whenever $n \geq m$.

Proposition 2.1: The limit of a sequence $\{\mathcal{U}_{\lambda,n}^x\}_n$ in a soft metric space $(\tilde{\mathcal{U}}, d)$, if it exists, is unique.

Definition 2.5: A sequence $\{\mathcal{U}_{\lambda,n}^x\}_n$ of soft points in a soft metric space $(\tilde{\mathcal{U}}, d)$ is said to be a Cauchy sequence in $(\tilde{\mathcal{U}}, d)$ if, for each $\hat{\epsilon} \tilde{>} \overline{0}$, there exists an $m \in \mathbb{N}$ such that $d(\mathcal{U}_{\lambda,i}^x, \mathcal{U}_{\lambda,j}^x) \tilde{<} \hat{\epsilon}$, for all $i, j \geq m$. That is, $d(\mathcal{U}_{\lambda,i}^x, \mathcal{U}_{\lambda,j}^x) \to \overline{0}$ as $i, j \to \infty$.

Proposition 2.2: Every convergent sequence $\{\mathcal{U}_{\lambda,n}^x\}_n$ in a soft metric space $(\tilde{\mathcal{U}}, d)$ is a Cauchy sequence.

Definition 2.6: A soft metric space $(\tilde{\mathcal{U}}, d)$ is called complete if every Cauchy sequence in $(\tilde{\mathcal{U}}, d)$ converges to some point of $\tilde{\mathcal{U}}$. In this case, we say that the soft metric d is complete.

If $(\tilde{\mathcal{U}}, d, A)$ is a soft metric space with A a (nonempty) finite set, then d induces in a natural way a compatible metric on $SP(\tilde{\mathcal{U}})$.

Theorem 2.1: Let $(\tilde{\mathcal{U}}, d, A)$ be a soft metric space with A a finite set. Define a function $m_d: SP(\tilde{\mathcal{U}}) \times SP(\tilde{\mathcal{U}}) \to \mathbb{R}^*$ as

$$m_d(\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y) = \max_{\eta \in A} d(\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y)(\eta)$$

for all $\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y \in SP(\tilde{\mathcal{U}})$. Then the following hold:

(1) m_d is a metric on $SP(\tilde{\mathcal{U}})$.

(2) For any sequence $\{\mathcal{U}_{\lambda,n}^x\}_n$ of soft points and a soft point $\mathcal{U}_\lambda^y$, we have

(2a) $\{\mathcal{U}_{\lambda,n}^x\}_n$ is a Cauchy sequence in $(\tilde{\mathcal{U}}, d, A)$ if and only if it is a Cauchy sequence in $(SP(\tilde{\mathcal{U}}), m_d)$.

(2b) $d(\mathcal{U}_\lambda^y, \mathcal{U}_{\lambda,n}^x) \to \overline{0}$ if and only if $m_d(\mathcal{U}_\lambda^y, \mathcal{U}_{\lambda,n}^x) \to 0$.

(3) $(\tilde{\mathcal{U}}, d, A)$ is complete if and only if $(SP(\tilde{\mathcal{U}}), m_d)$ is complete.

Theorem 2.2: [2] Let $(\tilde{\mathcal{U}}, d, A)$ be a complete soft metric space with A a finite set. Suppose that the soft mapping $f: \tilde{\mathcal{U}} \rightrightarrows \tilde{\mathcal{U}}$ satisfies

$$d\left(f(\mathcal{U}_\lambda^x), f(\mathcal{U}_\mu^y)\right) \tilde{\leq} \overline{c} d(\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y)$$

for all $\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y \in SP(\tilde{\mathcal{U}})$, where $\overline{0} \tilde{\leq} \overline{c} \tilde{<} \overline{1}$. Then f has a unique fixed point, i.e., there is a unique soft point $\mathcal{U}_\lambda^x$ such that $f(\mathcal{U}_\lambda^x) = \mathcal{U}_\lambda^x$.

Remark: Example 3.22 of [1] shows that condition 'A is a finite set' cannot be replaced with 'A is a countable set.'

Our next result provides a soft metric generalization of the celebrated Kannan fixed point theorem [40].

Theorem 2.3: [2] Let $(\tilde{\mathcal{U}}, d, A)$ be a complete soft metric space with A a finite set. Suppose that the soft mapping $f: \tilde{\mathcal{U}} \rightrightarrows \tilde{\mathcal{U}}$ satisfies

$$d\left(f(\mathcal{U}_\lambda^x), f(\mathcal{U}_\mu^y)\right) \tilde{\leq} \overline{c}\{d(\mathcal{U}_\lambda^x, f(\mathcal{U}_\lambda^x)) + d(\mathcal{U}_\mu^y, f(\mathcal{U}_\mu^y))\}$$

for all $\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y \in SP(\tilde{\mathcal{U}})$, where $\overline{0} \tilde{\leq} \overline{c} \tilde{<} \frac{\overline{1}}{2}$. Then f has a unique fixed point.

3. Convergence of Picard Iteration in soft metric space

Theorem 3.1: Let $(\tilde{\mathcal{U}}, d, A)$ be a complete soft metric space with A a finite set. Suppose that $f: \tilde{\mathcal{U}} \rightrightarrows \tilde{\mathcal{U}}$ satisfies

$$d\left(f(\mathcal{U}_\lambda^x), f(\mathcal{U}_\mu^y)\right) \tilde{\leq} \overline{c} d(\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y)$$

for all $\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y \in SP(\tilde{\mathcal{U}})$ where $\overline{0} \tilde{\leq} \overline{c} \tilde{<} \overline{1}$. Then the Picard iteration associated to f, i.e., the sequence $\{\mathcal{U}_{\lambda,n}^x\}_{n=0}^\infty$ of soft points in a soft metric space $(\tilde{\mathcal{U}}, d, A)$ defined by

$$\begin{aligned} \mathcal{U}_{\lambda,n}^x &= f(\mathcal{U}_{\lambda,n-1}^x) \\ &= f^n(\mathcal{U}_{\lambda,0}^x);\ n = 1,2,3, \ldots \ldots \end{aligned}$$

converges to its fixed point $\mathcal{U}_{\lambda_0}^{x_0}$, i.e., there is a soft point $\mathcal{U}_{\lambda_0}^{x_0}$ such that $f\left(\mathcal{U}_{\lambda_0}^{x_0}\right) = \mathcal{U}_{\lambda_0}^{x_0}$.

Proof: As we know from theorem 2 [2], the soft mapping f has a unique fixed point.
Now, we will show that for any given $\mathcal{U}_\lambda^x \in SP(\tilde{\mathcal{U}})$ (equivalently $\mathcal{U}_\lambda^x \in \tilde{\mathcal{U}}$) the Picard iteration $\mathcal{U}_{\lambda,n}^x$ is a Cauchy sequence.

The restriction of f to $SP(\tilde{\mathcal{U}})$ is a self-mapping on $SP(\tilde{\mathcal{U}})$ also denoted by f.

Note also that the real number c generates the constant soft real number $\overline{c}$ satisfies $0 \leq c < 1$. Then for each $\mathcal{U}_{\lambda,n}^x \in SP(\tilde{\mathcal{U}})$.

$$\begin{aligned} m_d\left(f(\mathcal{U}_{\lambda,1}^x), f(\mathcal{U}_{\lambda,0}^x)\right) &= \max_{\eta \in A} d(\mathcal{U}_{\lambda,1}^x, \mathcal{U}_{\lambda,0}^x)(\eta) \leq \max_{\eta \in A} c\left(d(\mathcal{U}_{\lambda,1}^x, \mathcal{U}_{\lambda,0}^x)(\eta)\right) \\ &= c\left[max_{\eta \in A}\, d(\mathcal{U}_{\lambda,1}^x, \mathcal{U}_{\lambda,0}^x)(\eta)\right] \\ &= c m_d(\mathcal{U}_{\lambda,1}^x, \mathcal{U}_{\lambda,0}^x) \end{aligned}$$

$$m_d\left(f(\mathcal{U}_{\lambda,1}^x), f(\mathcal{U}_{\lambda,0}^x)\right) \leq c m_d(\mathcal{U}_{\lambda,1}^x, \mathcal{U}_{\lambda,0}^x)$$

By induction

$$m_d\left(f(\mathcal{U}_{\lambda,n+1}^x), f(\mathcal{U}_{\lambda,n}^x)\right) \leq c^n m_d(\mathcal{U}_{\lambda,1}^x, \mathcal{U}_{\lambda,0}^x) \qquad ; n = 1,2,3, \ldots \ldots$$

Thus for any number $n, p \in \mathbb{N}, p > 0$, we have

$$\begin{aligned} m_d(\mathcal{U}_{\lambda,n+p}^x, \mathcal{U}_{\lambda,n}^x) &\leq \sum_{k=n}^{n+p-1} m_d(\mathcal{U}_{\lambda,k+1}^x, \mathcal{U}_{\lambda,k}^x) \\ &\leq \sum_{k=n}^{n+p-1} c^k m_d(\mathcal{U}_{\lambda,1}^x, \mathcal{U}_{\lambda,0}^x) \\ &\leq \frac{c^n}{1-c} m_d(\mathcal{U}_{\lambda,1}^x, \mathcal{U}_{\lambda,0}^x) \end{aligned} \tag{2}$$

Since $0 \leq c < 1$ it results that $c^n \to 0$ as $n \to \infty$. Equation (1) shows that $\mathcal{U}_{\lambda,n}^x$ is a Cauchy sequence. But $(\tilde{\mathcal{U}}, d, A)$ be a complete soft metric space if $\left(SP(\tilde{\mathcal{U}}), m_d\right)$ is complete by theorem 1[1].

Therefore $\mathcal{U}_{\lambda,n}^{x}$ converges to some $\mathcal{U}_{\lambda_0}^{x_0} \in SP(\tilde{\mathcal{U}})$ or $\mathcal{U}_{\lambda,n}^{x}$ converges to some $\mathcal{U}_{\lambda_0}^{x_0} \in \tilde{\mathcal{U}}$. On the other hand any soft Lipschitzian mapping us continuous so denoting

$$\lim_{n\to\infty} \mathcal{U}_{\lambda,n}^{x} = \mathcal{U}_{\lambda_0}^{x_0}$$

We find

$$\begin{aligned}\mathcal{U}_{\lambda_0}^{x_0} &= \lim_{n\to\infty} \mathcal{U}_{\lambda,n+1}^{x} \\ &= \lim_{n\to\infty} f(\mathcal{U}_{\lambda,n}^{x}) \\ &= f\left(\lim_{n\to\infty} \mathcal{U}_{\lambda,n}^{x}\right) \\ &= f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\end{aligned}$$

which gives $\mathcal{U}_{\lambda_0}^{x_0} = f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)$. i.e., $\mathcal{U}_{\lambda_0}^{x_0}$ is a soft fixed point of f.

This shows that for any $\mathcal{U}_{\lambda_0}^{x_0} \in \tilde{\mathcal{U}}$ the Picard iteration converges in $\tilde{\mathcal{U}}$ and its limit is a fixed point of f.

4. Picard operator in soft metric space

Definition 4.1: Let $(\tilde{\mathcal{U}}, d, A)$ be a complete soft metric space. A soft mapping $f: \tilde{\mathcal{U}} \rightrightarrows \tilde{\mathcal{U}}$ is called (strict) Picard mapping if there exists $\mathcal{U}_{\lambda_0}^{x_0} \in \tilde{\mathcal{U}}$ (or $\mathcal{U}_{\lambda_0}^{x_0} \in SP(\tilde{\mathcal{U}})$) such that $F_T = \{\mathcal{U}_{\lambda_0}^{x_0}\}$ and $f^n(\mathcal{U}_{\lambda,i}^{x}) = \mathcal{U}_{\lambda_0}^{x_0}$ for all $\mathcal{U}_{\lambda,i}^{x} \tilde{\in} \tilde{\mathcal{U}}$.

Example: If $(\tilde{\mathcal{U}}, d, A)$ be a complete soft metric space then any soft contraction as defined in theorem 2[], $f: \tilde{\mathcal{U}} \to \tilde{\mathcal{U}}$ is a Picard mapping.

Theorem 4.1: Let $(\tilde{\mathcal{U}}, d, A)$ be a complete soft metric space and the soft mapping $f: \tilde{\mathcal{U}} \to \tilde{\mathcal{U}}$ satisfies

$$d\left(f(\mathcal{U}_{\lambda}^{x}), f(\mathcal{U}_{\mu}^{y})\right) \tilde{\leqq} \bar{c}\{d(\mathcal{U}_{\lambda}^{x}, f(\mathcal{U}_{\lambda}^{x})) + d(\mathcal{U}_{\mu}^{y}, f(\mathcal{U}_{\mu}^{y}))\} \tag{3}$$

for all $\mathcal{U}_{\lambda}^{x}, \mathcal{U}_{\mu}^{y} \in SP(\tilde{\mathcal{U}})$, where $\bar{0} \tilde{\leqq} \bar{c} \tilde{<} \frac{\bar{1}}{2}$. Then f is a Picard operator.

Proof: Since $(\tilde{\mathcal{U}}, d, A)$ is a complete soft metric space. It follows from theorem 1[2] that $\left(SP(\tilde{\mathcal{U}}), m_d\right)$ is a complete metric space. Moreover, the restriction of f to $SP(\tilde{\mathcal{U}})$ is a soft mapping on $SP(\tilde{\mathcal{U}})$. Note also that a real number c generating a soft real number $\bar{c}$ satisfies $0 \le c < \frac{1}{2}$.

Let $\mathcal{U}_{\lambda,0}^{x} \in SP(\tilde{\mathcal{U}})$ and $\mathcal{U}_{\lambda,n}^{x} = f^n(\mathcal{U}_{\lambda,0}^{x})$; $n = 0,1,2,3, \dots \dots$ be the Picard iteration, then by equation (3) we have

$$\begin{aligned}m_d(\mathcal{U}_{\lambda,n}^{x}, \mathcal{U}_{\lambda,n+1}^{x}) &= m_d\left(f(\mathcal{U}_{\lambda,n-1}^{x}), f(\mathcal{U}_{\lambda,n}^{x})\right) \\ &= \max_{\eta\in A} d\left(f(\mathcal{U}_{\lambda,n-1}^{x}), f(\mathcal{U}_{\lambda,n}^{x})\right)(\eta) \\ &\le \max_{\eta\in A} c\left(d\left(\mathcal{U}_{\lambda,n-1}^{x}, f(\mathcal{U}_{\lambda,n-1}^{x})\right) + d\left(\mathcal{U}_{\lambda,n}^{x}, f(\mathcal{U}_{\lambda,n}^{x})\right)\right)(\eta) \\ &\le c\left\{m_d\left(\mathcal{U}_{\lambda,n-1}^{x}, f(\mathcal{U}_{\lambda,n-1}^{x})\right) + m_d\left(\mathcal{U}_{\lambda,n}^{x}, f(\mathcal{U}_{\lambda,n}^{x})\right)\right\} \\ &= c\{m_d(\mathcal{U}_{\lambda,n-1}^{x}, \mathcal{U}_{\lambda,n}^{x}) + m_d(\mathcal{U}_{\lambda,n}^{x}, \mathcal{U}_{\lambda,n+1}^{x})\}\end{aligned}$$

$$m_d(\mathcal{U}_{\lambda,n}^{x}, \mathcal{U}_{\lambda,n+1}^{x}) \le c\{m_d(\mathcal{U}_{\lambda,n-1}^{x}, \mathcal{U}_{\lambda,n}^{x}) + m_d(\mathcal{U}_{\lambda,n}^{x}, \mathcal{U}_{\lambda,n+1}^{x})\}$$

i.e., $$m_d(\mathcal{U}_{\lambda,n}^{x}, \mathcal{U}_{\lambda,n+1}^{x}) \le \frac{c}{1-c} m_d(\mathcal{U}_{\lambda,n-1}^{x}, \mathcal{U}_{\lambda,n}^{x}) \tag{4}$$

Since $0 \le \frac{c}{1-c} \le 1$ for $c \in \left[0, \frac{1}{2}\right)$, therefore $\{\mathcal{U}_{\lambda,n}^{x}\}$ is a Cauchy sequence in $(SP(\tilde{\mathcal{U}}), m_d)$; it is Cauchy sequence in $(\tilde{\mathcal{U}}, d, A)$ by Theorem 1[2]. Let $\mathcal{U}_{\lambda_0}^{x_0} \in SP(\tilde{\mathcal{U}})$ be its limit, then, we have $(SP(\tilde{\mathcal{U}}), m_d)$ is a complete metric space, therefore

$$m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\right) \le m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\lambda,n}^{x}\right) + m_d\left(\mathcal{U}_{\lambda,n}^{x}, f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\right)$$

$$= \max_{\eta \in A}\left\{d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\lambda,n}^{x}\right) + d\left(\mathcal{U}_{\lambda,n}^{x}, f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\right)\right\}(\eta)$$

$$= \max_{\eta \in A} d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\lambda,n}^{x}\right)(\eta) + \max_{\eta \in A} d\left(\mathcal{U}_{\lambda,n}^{x}, f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\right)(\eta)$$

$$\le \max_{\eta \in A} d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\lambda,n}^{x}\right)(\eta) + \max_{\eta \in A} d\left(f(\mathcal{U}_{\lambda,n-1}^{x}), f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\right)(\eta)$$

$$\le \max_{\eta \in A} d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\lambda,n}^{x}\right)(\eta) + \max_{\eta \in A} c\left\{d\left(\mathcal{U}_{\lambda,n-1}^{x}, f(\mathcal{U}_{\lambda,n-1}^{x})\right) + d\left(\mathcal{U}_{\lambda}^{x}, f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\right)\right\}(\eta)$$

$$\le m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\lambda,n}^{x}\right) + c m_d\left(\mathcal{U}_{\lambda,n-1}^{x}, f(\mathcal{U}_{\lambda,n-1}^{x})\right) + c m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\right)$$

$$= m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\lambda,n}^{x}\right) + c\{m_d(\mathcal{U}_{\lambda,n-1}^{x}, \mathcal{U}_{\lambda,n}^{x}) + m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\right)\}$$

$$(1-c) m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\right) \le m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\lambda,n}^{x}\right) + c m_d(\mathcal{U}_{\lambda,n-1}^{x}, \mathcal{U}_{\lambda,n}^{x})$$

$$m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\right) \le \frac{1}{1-c} m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\lambda,n}^{x}\right) + \frac{c}{1-c} m_d(\mathcal{U}_{\lambda,n-1}^{x}, \mathcal{U}_{\lambda,n}^{x})$$

From inequality (4) we have,

$$m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\right) \le \frac{1}{1-c} m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\lambda,n}^{x}\right) + \frac{c}{1-c} m_d(\mathcal{U}_{\lambda,n-1}^{x}, \mathcal{U}_{\lambda,n}^{x})$$

$$\le \frac{1}{1-c} m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\lambda,n}^{x}\right) + \left(\frac{c}{1-c}\right)^n m_d(\mathcal{U}_{\lambda,0}^{x}, \mathcal{U}_{\lambda,1}^{x}) \tag{5}$$

Now letting $n \to \infty$, we obtain

$$m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\right) = 0 \Leftrightarrow \mathcal{U}_{\lambda_0}^{x_0} = f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)$$

That is $F_T = \mathcal{U}_{\lambda_0}^{x_0}$ and therefore $\mathcal{U}_{\lambda,n}^{x} \to \mathcal{U}_{\lambda_0}^{x_0}$ as $n \to \infty$ for each $\mathcal{U}_{\lambda,0}^{x} \in SP(\tilde{\mathcal{U}})$.

Example 1: Let $\mathcal{U} = A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. The mapping $d: SP(\tilde{\mathcal{U}}) \times SP(\tilde{\mathcal{U}}) \to \mathbb{R}(A)^*$ given by $d(\mathcal{U}_{\lambda}^{x}, \mathcal{U}_{\mu}^{y}) = |\bar{x} - \bar{y}| + |\bar{\lambda} - \bar{\mu}|$ for all $\mathcal{U}_{\lambda}^{x}, \mathcal{U}_{\mu}^{y} \in SP(\tilde{\mathcal{U}})$, where $|.|$ denotes the modulus of soft real numbers in a soft metric space [see [10]]. Furthermore, as in [1] example (3.21) the soft metric space $(\tilde{\mathcal{U}}, d)$ is complete.

Let $f: \tilde{\mathcal{U}} \to \tilde{\mathcal{U}}$ such that $f(\mathcal{U}_{\lambda}^{x}) = \mathcal{U}_{1}^{\frac{x}{5}}$ for all $x \in \mathcal{U}, \lambda \in A$.

Given x ∈ [0, 2),y ∈ [2, ∞) and λ, μ ∈ A, for each η ∈ A, we have

$$d\left(f(\mathcal{U}_{\lambda}^{x}), f(\mathcal{U}_{\mu}^{y})\right) = d\left(\mathcal{U}_{1}^{\frac{x}{5}}, \mathcal{U}_{1}^{\frac{y}{5}}\right)(\eta)$$

$$= \left|\frac{x}{5} - \frac{y}{5}\right|$$

$$= \frac{1}{5}|x - y|$$

$$\le \frac{1}{5}|x + y|$$

$$\leq \frac{1}{4}\left(\left(x-\frac{x}{5}\right)+\left(y-\frac{y}{5}\right)\right)$$
$$\leq \frac{1}{4}\left\{d\left(\mathcal{U}_\lambda^x, f(\mathcal{U}_\lambda^x)\right)+d\left(\mathcal{U}_\mu^y, f\left(\mathcal{U}_\mu^y\right)\right)\right\}$$
$$d\left(f(\mathcal{U}_\lambda^x), f\left(\mathcal{U}_\mu^y\right)\right) \leq \frac{1}{4}\left\{d\left(\mathcal{U}_\lambda^x, f(\mathcal{U}_\lambda^x)\right)+d\left(\mathcal{U}_\mu^y, f\left(\mathcal{U}_\mu^y\right)\right)\right\}$$

Then f is a Picard operator. However, f has no fixed point because A is not finite. Hence theorem 1 does not hold.

Example 2: Let $\mathcal{U}=\mathbb{R}^+$ and $A=\{0,1\}$, from [10] Example (4.3) the mapping
$$d: SP(\tilde{\mathcal{U}}) \times SP(\tilde{\mathcal{U}}) \to \mathbb{R}(A)^*$$
given by
$$d\left(\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y\right)=|\overline{x}-\overline{y}|+|\overline{\lambda}-\overline{\mu}|$$
for all $\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y \in SP(\tilde{\mathcal{U}})$, is a soft metric on $\tilde{\mathcal{U}}$. Since $\mathbb{R}^+$ is complete for the Euclidean metric, we deduce that $(\tilde{\mathcal{U}}, d)$ is a complete soft metric space. Let $f: \tilde{\mathcal{U}} \to \tilde{\mathcal{U}}$ such that $f(\mathcal{U}_0^x)=f(\mathcal{U}_1^x)=\mathcal{U}_0^0$ if $x \in [0,2)$ and $f(\mathcal{U}_0^x)=f(\mathcal{U}_1^x)=\mathcal{U}_0^{\frac{1}{2}}$ if $x \in [2,\infty)$. Let $\overline{c}$ be a constant soft real number such that $\overline{0} \cong \overline{c} \lesssim \overline{1}$. Then there is real number $c \in [0,1)$ such that $c=\overline{c}(\eta)$ for all $\eta \in A$. Choose $z \in [0,2)$ such that $c(2-z)<\frac{1}{2}$. Then for each $\eta \in A$, we have

$$\begin{aligned} d\left(f(\mathcal{U}_0^x), f(\mathcal{U}_0^z)\right) &= d\left(\mathcal{U}_0^{\frac{1}{2}}, \mathcal{U}_0^0\right)(\eta) \\ &= \left\|\frac{1}{2}-0\right\|+\|0-0\| \\ &= \frac{1}{2} \\ &= c(2-z) \\ &= cd(\mathcal{U}_0^2, \mathcal{U}_0^z)(\eta) \end{aligned}$$

Therefore f does not satisfy the condition (1.1) of theorem [1] for any $\overline{0} \lesssim \overline{c} \lesssim \overline{1}$.

$$\begin{aligned} d\left(f(\mathcal{U}_0^x), f(\mathcal{U}_0^z)\right) &= \frac{1}{2} \\ &= \frac{1}{3}\left(2-\frac{1}{2}\right) \\ &\leq \frac{1}{3}\left(x+z-\frac{1}{2}\right) \\ &= \frac{1}{3}\left(d(\mathcal{U}_0^x, \mathcal{U}_0^0)+d\left(\mathcal{U}_0^z, \mathcal{U}_0^{\frac{1}{2}}\right)\right)(\eta) \end{aligned}$$

Let $f: \tilde{\mathcal{U}} \to \tilde{\mathcal{U}}$ be such that $f(\mathcal{U}_\lambda^x)=\mathcal{U}_0^0$ if $x \in [0,2)$ and $f\left(\mathcal{U}_\mu^y\right)=\mathcal{U}_0^{\frac{1}{2}}$ if $x \in [2,\infty)$.

However taking without loss of generality $x \in [0,2)$ and $y \in [2,\infty)$, we obtain for $\lambda, \mu, \nu \in A$

$$\begin{aligned} d\left(f(\mathcal{U}_\lambda^x), f\left(\mathcal{U}_\mu^z\right)\right)(\eta) &= d\left(\mathcal{U}_0^0, \mathcal{U}_0^{\frac{1}{2}}\right)(\eta) \\ &\leq \frac{1}{3}\left(d(\mathcal{U}_\lambda^x, \mathcal{U}_0^0)+d\left(\mathcal{U}_\mu^y, \mathcal{U}_0^{\frac{1}{2}}\right)\right)(\eta) \end{aligned}$$

Therefore f satisfies inequality (3) of theorem (4.1) for $\overline{c}=\frac{1}{3}$. In fact, $\mathcal{U}_0^0$ is the unique fixed point of f, therefore, f is a Picard operator.

Theorem 4.2: Let $(\tilde{\mathcal{U}}, d, A)$ be a complete soft metric space and $f: \tilde{\mathcal{U}} \to \tilde{\mathcal{U}}$ be a soft mapping for which there exists soft real numbers α, β and γ satisfying $\overline{0} \cong\!\!\!\!\!\!< \overline{\alpha} \,\tilde{<}\, \overline{\tfrac{1}{2}}$, $\overline{0} \,\tilde{\leq}\, \overline{\beta} \,\tilde{<}\, \overline{\tfrac{1}{2}}$ and $\overline{0} \,\tilde{\leq}\, \overline{\gamma} \,\tilde{<}\, \overline{\tfrac{1}{2}}$ such that for each $\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y \in SP(\tilde{\mathcal{U}})$ at least one of the following is true:

(1) $d\left(f(\mathcal{U}_\lambda^x), f(\mathcal{U}_\mu^y)\right) \,\tilde{\leq}\, \overline{\alpha} d(\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y)$

(2) $d\left(f(\mathcal{U}_\lambda^x), f(\mathcal{U}_\mu^y)\right) \,\tilde{\leq}\, \overline{\beta}\left[d(\mathcal{U}_\lambda^x, f(\mathcal{U}_\lambda^x)) + d\left(\mathcal{U}_\mu^y, f(\mathcal{U}_\mu^y)\right)\right]$

(3) $d\left(f(\mathcal{U}_\lambda^x), f(\mathcal{U}_\mu^y)\right) \,\tilde{\leq}\, \overline{\gamma}\left[d\left(\mathcal{U}_\lambda^x, f(\mathcal{U}_\mu^y)\right) + d\left(\mathcal{U}_\mu^y, f(\mathcal{U}_\lambda^x)\right)\right]$.

Then f is a Picard operator.

Proof: We first consider $\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y \in SP(\tilde{\mathcal{U}})$. At least one of z_1, z_2 or z_3 is true. Note also that real numbers α, β, γ generate a soft real number $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$ satisfies $0 \leq \alpha < \frac{1}{2}$, $0 \leq \beta < \frac{1}{2}$, $0 \leq \gamma < \frac{1}{2}$. If z_2 holds then we have

$$\begin{aligned} m_d\left(f(\mathcal{U}_\lambda^x), f(\mathcal{U}_\mu^y)\right) &= \max\nolimits_{\eta \in A} d\left(f(\mathcal{U}_\lambda^x), f(\mathcal{U}_\mu^y)\right)(\eta) \\ &\leq \max\nolimits_{\eta \in \beta}\left[d(\mathcal{U}_\lambda^x, f(\mathcal{U}_\lambda^x)) + d\left(\mathcal{U}_\mu^y, f(\mathcal{U}_\mu^y)\right)\right](\eta) \\ &= \beta\left[m_d(\mathcal{U}_\lambda^x, f(\mathcal{U}_\lambda^x)) + m_d\left(\mathcal{U}_\mu^y, f(\mathcal{U}_\lambda^x)\right)\right](\eta) \\ &\leq \beta\left[m_d(\mathcal{U}_\lambda^x, f(\mathcal{U}_\lambda^x)) + m_d(\mathcal{U}_\mu^y, \mathcal{U}_\lambda^x) + m_d(\mathcal{U}_\lambda^x, f(\mathcal{U}_\lambda^x)) + m_d\left(f(\mathcal{U}_\lambda^x), f(\mathcal{U}_\mu^y)\right)\right] \end{aligned}$$

Since m_d is a metric on $SP(\tilde{\mathcal{U}})$, by theorem 1[2]

$$(1-\beta) m_d\left(f(\mathcal{U}_\lambda^x), f(\mathcal{U}_\mu^y)\right) \leq 2\beta m_d(\mathcal{U}_\lambda^x, f(\mathcal{U}_\lambda^x)) + \beta m_d(\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y)$$

$$m_d\left(f(\mathcal{U}_\lambda^x), f(\mathcal{U}_\mu^y)\right) \leq \frac{2\beta}{1-\beta} m_d(\mathcal{U}_\lambda^x, f(\mathcal{U}_\lambda^x)) + \frac{\beta}{1-\beta} m_d(\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y)$$

If z_3 holds then similarly

$$m_d\left(f(\mathcal{U}_\lambda^x), f(\mathcal{U}_\mu^y)\right) \leq \frac{2\gamma}{1-\gamma} m_d(\mathcal{U}_\lambda^x, f(\mathcal{U}_\lambda^x)) + \frac{\gamma}{1-\gamma} m_d(\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y)$$

Therefore denoting

$$\delta = \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$$

We have $0 \leq \delta < 1$ and then for all $\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y \in SP(\tilde{\mathcal{U}})$, the following inequality

$$m_d\left(f(\mathcal{U}_\lambda^x), f(\mathcal{U}_\mu^y)\right) \leq 2\delta m_d(\mathcal{U}_\lambda^x, f(\mathcal{U}_\lambda^x)) + \delta m_d(\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y) \tag{6}$$

valid for all $\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y \in SP(\tilde{\mathcal{U}})$.

In similar manner, we obtain

$$m_d\left(f(\mathcal{U}_\lambda^x), f(\mathcal{U}_\mu^y)\right) \leq 2\delta m_d\left(\mathcal{U}_\mu^y, f(\mathcal{U}_\mu^y)\right) + \delta m_d(\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y) \quad (7)$$

for all $\mathcal{U}_\lambda^x, \mathcal{U}_\mu^y \in SP(\tilde{\mathcal{U}})$.

Now we will show that f has unique fixed point.

Let $\mathcal{U}_{\lambda,0}^x \in SP(\tilde{\mathcal{U}})$ be arbitrary and $\{\mathcal{U}_{\lambda,n}^x\}_n$

$$\mathcal{U}_{\lambda,n}^x = f^n(\mathcal{U}_{\lambda,0}^x) \qquad ; n = 0,1,2, \ldots \ldots$$

be the Picard iteration associated to f.

If $\mathcal{U}_{\lambda,n}^{x}$ and $\mathcal{U}_{\lambda,n-1}^{x}$ are two successive approximations, then by (7)

$$m_d\left(f(\mathcal{U}_{\lambda,n-1}^{x}), f(\mathcal{U}_{\lambda,n}^{x})\right) \leq 2\delta m_d\left(\mathcal{U}_{\lambda,n}^{x}, f(\mathcal{U}_{\lambda,n}^{x})\right) + \delta m_d(\mathcal{U}_{\lambda,n-1}^{x}, \mathcal{U}_{\lambda,n}^{x})$$

$$m_d(\mathcal{U}_{\lambda,n}^{x}, \mathcal{U}_{\lambda,n+1}^{x}) \leq 2\delta m_d(\mathcal{U}_{\lambda,n}^{x}, \mathcal{U}_{\lambda,n+1}^{x}) + \delta m_d(\mathcal{U}_{\lambda,n-1}^{x}, \mathcal{U}_{\lambda,n}^{x})$$

$$m_d(\mathcal{U}_{\lambda,n}^{x}, \mathcal{U}_{\lambda,n+1}^{x}) \leq \frac{\delta}{1-2\delta} m_d(\mathcal{U}_{\lambda,n-1}^{x}, \mathcal{U}_{\lambda,n}^{x})$$

$$< \delta m_d(\mathcal{U}_{\lambda,n-1}^{x}, \mathcal{U}_{\lambda,n}^{x})$$

From this we deduce that $\{\mathcal{U}_{\lambda,n}^{x}\}$ is a Cauchy sequence, hence convergent too.

Let $\mathcal{U}_{\lambda}^{x} \in SP(\tilde{\mathcal{U}})$ be its limit. In particular, we have

$\lim_{n\to\infty} m_d(\mathcal{U}_{\lambda,n+1}^{x}, \mathcal{U}_{\lambda,n}^{x}) = 0$ (8)

From (6) we get that (as m_d is a metric on $SP(\tilde{\mathcal{U}})$)

$$m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\right) \leq m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\lambda,n+1}^{x}\right) + m_d\left(\mathcal{U}_{\lambda,n+1}^{x}, f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\right)$$

$$= m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\lambda,n+1}^{x}\right) + m_d\left(f(\mathcal{U}_{\lambda,n}^{x}), f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\right)$$

$$= m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, f(\mathcal{U}_{\lambda,n}^{x})\right) + m_d\left(f(\mathcal{U}_{\lambda,n}^{x}), f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\right)$$

$$\leq m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, f(\mathcal{U}_{\lambda,n}^{x})\right) + \delta m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\lambda,n}^{x}\right) + 2\delta m_d\left(\mathcal{U}_{\lambda,n}^{x}, f(\mathcal{U}_{\lambda,n}^{x})\right)$$

Which by letting $n \to \infty$ yields

$$m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\right) = 0 \Leftrightarrow \mathcal{U}_{\lambda_0}^{x_0} = f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)$$

Since $m_d\left(\mathcal{U}_{\lambda,n}^{x}, f(\mathcal{U}_{\lambda,n}^{x})\right) = m_d(\mathcal{U}_{\lambda,n}^{x}, \mathcal{U}_{\lambda,n+1}^{x})$

From (8) we have $m_d\left(\mathcal{U}_{\lambda,n}^{x}, f(\mathcal{U}_{\lambda,n}^{x})\right) = 0$

And therefore

$$F_T = \{\mathcal{U}_{\lambda_0}^{x_0}\}$$

and $\mathcal{U}_{\lambda,n}^{x} \to \mathcal{U}_{\lambda_0}^{x_0}$ as $n \to \infty$ for each $\mathcal{U}_{\lambda,0}^{x} \in SP(\tilde{\mathcal{U}})$.

Therefore $\mathcal{U}_{\lambda,n}^{x} \to \mathcal{U}_{\lambda_0}^{x_0}$ as $n \to \infty$ for each $\mathcal{U}_{\lambda,0}^{x} \tilde{\in} \tilde{\mathcal{U}}$.

5. Weak Contraction

Definition 5.1: Let $(\tilde{\mathcal{U}}, d, A)$ be a soft metric space. A map $f: \tilde{\mathcal{U}} \to \tilde{\mathcal{U}}$ is called weak contraction if there exists a constant $\overline{0} \mathrel{\tilde{\leqq}} \tilde{\delta} \mathrel{\tilde{<}} \overline{1}$ and some $\overline{L} \mathrel{\tilde{\geqq}} \overline{0}$ such that

$d\left(f(\mathcal{U}_{\lambda}^{x}), f(\mathcal{U}_{\mu}^{y})\right) \mathrel{\tilde{\leqq}} \tilde{\delta} d(\mathcal{U}_{\lambda}^{x}, \mathcal{U}_{\mu}^{y}) + \overline{L} d(\mathcal{U}_{\mu}^{y}, f(\mathcal{U}_{\lambda}^{x}))$ (9)

For all $\mathcal{U}_{\lambda}^{x}, \mathcal{U}_{\mu}^{y} \in SP(\tilde{\mathcal{U}})$.

Proposition 5.1: Any Kannan mapping, i.e., any mapping satisfying the contractive condition in Theorem 3 [2] is a weak contraction.

Proof: Let $(\tilde{\mathcal{U}}, d, A)$ be a soft metric space. Also, the real number c generates the constant soft real number $\overline{c}$ satisfies $0 \leq c \leq \frac{1}{2}$

By property of soft metric space

$$d\left(\mathcal{U}_{\mu}^{y}, f(\mathcal{U}_{\lambda}^{x})\right)(\eta) \leq c\left[d\big(\mathcal{U}_{\lambda}^{x}, f(\mathcal{U}_{\lambda}^{x})\big) + d\left(\mathcal{U}_{\mu}^{y}, f\big(\mathcal{U}_{\mu}^{y}\big)\right)\right](\eta)$$
$$\leq c\left\{\left[d\left(\mathcal{U}_{\lambda}^{x}, f\big(\mathcal{U}_{\mu}^{y}\big)\right) + d\left(\mathcal{U}_{\mu}^{y}, f(\mathcal{U}_{\lambda}^{x})\right)\right] + \left[d\left(\mathcal{U}_{\mu}^{y}, f(\mathcal{U}_{\lambda}^{x})\right) + d\left(f(\mathcal{U}_{\lambda}^{x}), f\big(\mathcal{U}_{\mu}^{y}\big)\right)\right]\right\}(\eta)$$

which yields

$$(1-c)d\left(f(\mathcal{U}_{\lambda}^{x}), f\big(\mathcal{U}_{\mu}^{y}\big)\right)(\eta) \leq cd\big(\mathcal{U}_{\lambda}^{x}, \mathcal{U}_{\mu}^{y}\big)(\eta) + 2cd\left(\mathcal{U}_{\mu}^{y}, f(\mathcal{U}_{\lambda}^{x})\right)(\eta)$$

which implies

$$d\left(f(\mathcal{U}_{\lambda}^{x}), f\big(\mathcal{U}_{\mu}^{y}\big)\right) \leq \frac{c}{1-c}\, d\big(\mathcal{U}_{\lambda}^{x}, \mathcal{U}_{\mu}^{y}\big)(\eta) + \frac{2c}{1-c}\, d\big(\mathcal{U}_{\lambda}^{x}, f(\mathcal{U}_{\lambda}^{x})\big)(\eta) \qquad \text{for all } \mathcal{U}_{\lambda}^{x}, \mathcal{U}_{\mu}^{y} \in SP(\tilde{\mathcal{U}})$$

and hence in view of $0 \leq c \leq 1/2$ from (9) holds with $\delta = \frac{c}{1-c}$ and $L = \frac{2c}{1-c}$ .For corresponding real numbers δ and L we have a soft real number δ and L, we have a soft real number $\overline{\delta} = \frac{c}{1-c}$ and $\overline{L} = \frac{2c}{1-c}$. Hence, any Kannan mapping is a weak contraction.

Proposition 5.2: Any mapping f satisfying the contractive condition, there exists $\overline{0} \tilde{\leq} \overline{c} \tilde{<} \tilde{\frac{1}{2}}$ such that

$$d\left(f(\mathcal{U}_{\lambda}^{x}), f\big(\mathcal{U}_{\mu}^{y}\big)\right) \tilde{\leq} \tilde{c}\left[d\left(\mathcal{U}_{\lambda}^{x}, f\big(\mathcal{U}_{\mu}^{y}\big)\right) + d\left(\mathcal{U}_{\mu}^{y}, f(\mathcal{U}_{\lambda}^{x})\right)\right]$$

for all $\mathcal{U}_{\lambda}^{x}, \mathcal{U}_{\mu}^{y} \in SP(\tilde{\mathcal{U}})$ is a weak contraction.

Proof:

$$d\left(\mathcal{U}_{\lambda}^{x}, f\big(\mathcal{U}_{\mu}^{y}\big)\right)(\eta) \leq \left[d\big(\mathcal{U}_{\lambda}^{x}, \mathcal{U}_{\mu}^{y}\big) + d\left(\mathcal{U}_{\mu}^{y}, f(\mathcal{U}_{\lambda}^{x})\right) + d\left(f(\mathcal{U}_{\lambda}^{x}), f\big(\mathcal{U}_{\mu}^{y}\big)\right)\right](\eta)$$

Now

$$d\left(f(\mathcal{U}_{\lambda}^{x}), f\big(\mathcal{U}_{\mu}^{y}\big)\right)(\eta) \leq c\left[d\left(\mathcal{U}_{\lambda}^{x}, f\big(\mathcal{U}_{\mu}^{y}\big)\right) + d\left(\mathcal{U}_{\mu}^{y}, f(\mathcal{U}_{\lambda}^{x})\right)\right](\eta)$$
$$\leq c\left[d\big(\mathcal{U}_{\lambda}^{x}, \mathcal{U}_{\mu}^{y}\big) + d\left(\mathcal{U}_{\mu}^{y}, f(\mathcal{U}_{\lambda}^{x})\right) + d\left(f(\mathcal{U}_{\lambda}^{x}), f\big(\mathcal{U}_{\mu}^{y}\big)\right) + d\big(\mathcal{U}_{\mu}^{y}, f(\mathcal{U}_{\lambda}^{x})\big)\right](\eta)$$
$$d\left(f(\mathcal{U}_{\lambda}^{x}), f\big(\mathcal{U}_{\mu}^{y}\big)\right)(\eta) \leq \frac{c}{1-c} d\big(\mathcal{U}_{\lambda}^{x}, \mathcal{U}_{\mu}^{y}\big)(\eta) + \frac{2c}{1-c} d\left(\mathcal{U}_{\mu}^{y}, f(\mathcal{U}_{\lambda}^{x})\right)(\eta)$$

Since $c < \frac{1}{2}$ and $L = \frac{2c}{1-c} \geq 0$ with $\delta = \frac{c}{1-c} < 1$. This implies

$$d\left(f(\mathcal{U}_{\lambda}^{x}), f\big(\mathcal{U}_{\mu}^{y}\big)\right)(\eta) \leq \delta d\big(\mathcal{U}_{\lambda}^{x}, \mathcal{U}_{\mu}^{y}\big)(\eta) + \mathrm{L}d\left(\mathcal{U}_{\mu}^{y}, f(\mathcal{U}_{\lambda}^{x})\right)(\eta)$$

for any real numbers δ and L there is a soft real number $\bar{\delta}(\eta)$ and $\bar{L}(\eta)$ such that

$$d\left(f(\mathcal{U}_{\lambda}^{x}), f\big(\mathcal{U}_{\mu}^{y}\big)\right) \tilde{\leq} \bar{\delta} d\big(\mathcal{U}_{\lambda}^{x}, \mathcal{U}_{\mu}^{y}\big) + \bar{L} d\left(\mathcal{U}_{\mu}^{y}, f(\mathcal{U}_{\lambda}^{x})\right)$$
$$\overline{0} \tilde{<} \bar{\delta} \tilde{<} \overline{1} \quad \& \ \overline{L} \tilde{\geq} 0.$$

Theorem 5.1: Let $(\tilde{\mathcal{U}}, d, A)$ be a complete soft metric space and $f: \tilde{\mathcal{U}} \to \tilde{\mathcal{U}}$ be a weak contraction, i.e., a mapping satisfying (9) with $\overline{0} \tilde{<} \bar{\delta} \tilde{<} \overline{1}$ and some $\bar{L} \tilde{\geq} \overline{0}$. Then

1. Fix $|f| \neq \emptyset$.
2. For any $\mathcal{U}_{\lambda_0}^{x_0} \in X$, the Picard iteration $\{\mathcal{U}_{\lambda}^{x}\}_n$ given by

$$f\big(\mathcal{U}_{\lambda,n}^{x}\big) = \mathcal{U}_{\lambda,n+1}^{x} \tag{10}$$

converges to some $\mathcal{U}_{\lambda_0}^{x_0} \in \mathrm{Fix}(f)$.

Proof: We shall prove that f has at least one fixed point in $\tilde{U}$. Let $\mathcal{U}_{\lambda_0}^{x_0} \in \mathrm{SP}(\tilde{U})$ be arbitrary and let $\{\mathcal{U}_{\lambda,n}^{x}\}_{n=0}^{\infty}$ be the Picard iteration defined by

$$\mathcal{U}_{\lambda,n+1}^{x} = f(\mathcal{U}_{\lambda,n}^{x})$$

$$\max d\left(f(\mathcal{U}_{\lambda,n-1}^{x}), f(\mathcal{U}_{\lambda,n}^{x})\right)(\eta) \leq \max \delta d(\mathcal{U}_{\lambda,n-1}^{x}, \mathcal{U}_{\lambda,n}^{x})(\eta) + \max(Ld(\mathcal{U}_{\lambda,n}^{x}, f(\mathcal{U}_{\lambda,n-1}^{x})(\eta))$$

which shows that

$$m_d(\mathcal{U}_{\lambda,n}^{x}, \mathcal{U}_{\lambda,n+1}^{x}) \leq \delta m_d(\mathcal{U}_{\lambda,n-1}^{x}, \mathcal{U}_{\lambda,n}^{x}) + \mathrm{Lm}_d(\mathcal{U}_{\lambda,n}^{x}, \mathcal{U}_{\lambda,n}^{x})$$
$$m_d(\mathcal{U}_{\lambda,n}^{x}, \mathcal{U}_{\lambda,n+1}^{x}) \leq \delta m_d(\mathcal{U}_{\lambda,n-1}^{x}, \mathcal{U}_{\lambda,n}^{x})$$
$$\therefore\ m_d(\mathcal{U}_{\lambda,n}^{x}, \mathcal{U}_{\lambda,n+1}^{x}) \leq \delta^n m_d(\mathcal{U}_{\lambda,0}^{x}, \mathcal{U}_{\lambda,1}^{x})\ n = 0,1,2,3, \dots$$

and then

$$\begin{aligned} m_d(\mathcal{U}_{\lambda,n}^{x}, \mathcal{U}_{\lambda,n+p}^{x}) &\leq \delta^n(1 + \delta + \delta^2 + \cdots + \delta^{p-1}) m_d(\mathcal{U}_{\lambda,0}^{x}, \mathcal{U}_{\lambda,1}^{x}) \\ &= \frac{\delta^n}{1-\delta}(1 - \delta^p) m_d(\mathcal{U}_{\lambda,0}^{x}, \mathcal{U}_{\lambda,1}^{x}) \qquad n, p \in N, p \neq 0 \end{aligned} \tag{11}$$

Since a real number $0 < \delta < 1$ generates a soft real number $\bar{0} \tilde{<} \bar{\delta} \tilde{<} \bar{1}$. In inequality (11) shows that $\{\mathcal{U}_{\lambda,n}^{x}\}_n$ is a Cauchy sequence and hence is convergent. $\mathcal{U}_{\lambda_0}^{x_0} = \lim_{n\to\infty} \mathcal{U}_{\lambda,n}^{x}$ Then

$$\begin{aligned} m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\right) &= \max_{\eta\in A}\left\{d\left(\mathcal{U}_{\lambda_0}^{x_0}, f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\right)\right\}(\eta) \\ &\leq \max\left\{d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\lambda,n+1}^{x}\right) + d\left(f(\mathcal{U}_{\lambda,n}^{x}), f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\right)\right\}(\eta) \\ &\leq \max_{\eta\in A}\left\{d\left(\mathcal{U}_{\lambda_0}^{x_0}, f(\mathcal{U}_{\lambda,n}^{x})\right) + \delta d\left(\mathcal{U}_{\lambda,n}^{x}, \mathcal{U}_{\lambda_0}^{x_0}\right) + Ld\left(\mathcal{U}_{\lambda_0}^{x_0}, f(\mathcal{U}_{\lambda,n}^{x})\right)\right\}(\eta) \\ m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\right) &\leq (1 + L) m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, f(\mathcal{U}_{\lambda,n}^{x})\right) + \delta m_d\left(\mathcal{U}_{\lambda,n}^{x} \mathcal{U}_{\lambda_0}^{x_0}\right) \end{aligned} \tag{12}$$

valid for all $n \geq 0$. letting $n \to \infty$ in (12) we have

$$m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\right) = 0$$

$\therefore \mathcal{U}_{\lambda_0}^{x_0}$ is a fixed point of f.

Remark: The weak contractions need not have a unique fixed point. If possible to force the uniqueness of the fixed point of a weak contraction by imposing an additional contractive condition, quite similar to (3) as shown by the next theorem.

Theorem 5.2: Let $(\tilde{U}, d, A)$ be a complete soft metric space and $f: \tilde{U} \to \tilde{U}$ be a weak contraction for which there exists some $\bar{0} \tilde{<} \bar{\theta} \tilde{<} \bar{1}$ and $\overline{L_1} \tilde{\geq} \bar{0}$ such that

$$d\left(f\left(\mathcal{U}_{\lambda}^{x}, f(\mathcal{U}_{\mu}^{y}\right)\right) \tilde{\leq} \bar{\theta} d(\mathcal{U}_{\lambda}^{x}, \mathcal{U}_{\mu}^{y}) + \overline{L_1} d\left(\mathcal{U}_{\lambda}^{x}, f(\mathcal{U}_{\lambda,n}^{x})\right) \tag{13}$$

for all $\mathcal{U}_{\lambda}^{x}, \mathcal{U}_{\mu}^{y} \in \mathrm{SP}(\tilde{U})$ Then

1. f has a unique fixed point, i.e., $F(f) = \left\{\mathcal{U}_{\lambda_0}^{x_0}\right\}$
2. The Picard iteration $\{\mathcal{U}_{\lambda,n}^{x}\}_n$ given by (10) converges to $\mathcal{U}_{\lambda_0}^{x_0}$, for any $\mathcal{U}_{\lambda,0}^{x} \in \mathrm{SP}(\tilde{U})$

Proof: Assume f has two distinct fixed points $\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\mu_0}^{y_0} \in \mathrm{SP}(\tilde{U})$. Then by (13), we get

$$\max_{\eta}\left\{d\left(f\left(\mathcal{U}_{\lambda_0}^{x_0}\right), f(\mathcal{U}_{\mu_0}^{y_0})\right)\right\}(\eta) \leq \max_{\eta}\left\{\theta d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\mu_0}^{y_0}\right)\right\}(\eta) + \max_{\eta}\left\{L_1 d\left(\mathcal{U}_{\lambda_0}^{x_0}, f\left(\mathcal{U}_{\lambda_0}^{x_0}\right)\right)\right\}(\eta) \tag{14}$$

for any real number θ and L_1 there exists a soft real number $\bar{\theta}$ and $\overline{L_1}$ such that $0 < \theta < 1$ and $L_1 \geq 0$. Now

$$m_d\left(f\left(\mathcal{U}_{\lambda_0}^{x_0}\right), f(\mathcal{U}_{\mu_0}^{y_0})\right) \leq \theta m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\mu_0}^{y_0}\right) + L_1 m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\lambda_0}^{x_0}\right)$$

$$\Rightarrow \quad m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\mu_0}^{y_0}\right) \leq \theta m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\mu_0}^{y_0}\right)$$

$$\text{or } (1-\theta) m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\mu_0}^{y_0}\right) \leq 0$$

$$\text{so contradicting } m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\mu_0}^{y_0}\right) > 0$$

Hence

$$(1-\theta) m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\mu_0}^{y_0}\right) = 0$$

$$\Rightarrow \quad m_d\left(\mathcal{U}_{\lambda_0}^{x_0}, \mathcal{U}_{\mu_0}^{y_0}\right) = 0$$

$$\Rightarrow \quad \mathcal{U}_{\lambda_0}^{x_0} = \mathcal{U}_{\mu_0}^{y_0}$$

Hence f has a unique fixed point, i.e., $F(f) = \left\{\mathcal{U}_{\lambda_0}^{x_0}\right\}$.

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