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Optimization and Constraint Satisfaction of Production Function in Industrial Organisation

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Abstract

Recently businessmen, as well as industrialists, have been very much concerned about the theory of firm in order to make correct decisions regarding what item, how much and how, the quality of material and technological improvement also. The production function is formulated as a non-linear problem in the study whereby optimality is determined using KKT conditions and this approach has been demonstrated through theoretical analysis. These findings emphasize the significance of non-linearity in production models and thus contribute to better strategies of production management. The aim of this paper is to develop and use deterministic simulation models to analyze, improve, and optimize manufacturing processes in industrial organization. The purpose of this study is to enhance our understanding of production dynamics, as well as create deterministic simulation models for analyzing and optimizing production functions across industrial environments with an eye on fulfilling the constraint's satisfaction.

Keywords: Cobb-Douglas Production function, Marginal Cost, KKT, constraint satisfaction, CRS, DRS, IRS.

Mathematics Subject Classification: 65K10, 78M50

1. Introduction

Optimization is a useful technique for determining the optimal solution to a problem. In other words, optimization is the problem of choosing suitable inputs under given circumstances in order to get the best possible output. For instance, optimization can be used in production models to adjust different inputs and make them more effective in order to get the best output for the production of a particular industry. An optimization problem usually consists of three ingredients: an objective function, a set of constraints and a number of decision variables. In this scenario, once we have modeled a problem, it can be solved using the available optimization techniques to find the optimal solution. Production is a scientific process that involves converting desired goods and services into outputs by adding economic value to various inputs such as raw materials, energy, machinery, labor, money, and management. Decisions regarding the size of production are one of the most important decisions taken in an enterprise. An industrialist tries to plan, organize, direct and control the production system, which is strictly linked to the production management process. One of the problems of managing enterprises in difficult market conditions is to make timely and appropriate decisions in connection with ongoing changes in the economic situation. The production functions are positive noncontact functions that specify the output of a firm, an industry, or an entire economy for all combinations of inputs. Almost all economic theories presuppose a production function, either on the firm level or the aggregate level. Hence, the production function is one of the key concepts of mainstream neoclassical theories. By assuming that the maximum output technologically possible from a given

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set of inputs is achieved, economists always using a production function in the analysis are abstracting from the engineering and managerial problems inherently associated with a particular production process.

Decisions concerning the size of production is one of the most important decisions made in the industry. Economic and mathematical modeling methods may be ways to solve such problems. Wang and Fu [14] discuss the framework of production functions that allows assessing the productivity of resources used in the production process, forecasting economic growth, forming a modification of the production development scenario and optimization of the operation of the economic entity subject to a predetermined criterion and available resource constraints. The production function is a mathematical relation, determining the relation between the factors and the quantity of input for production and the number of goods it produces most efficiently. It answers the queries related to marginal productivity, level of production and chief cheapest mode of production of goods. Production functions lie in the basis of modeling the activities of various types of production structures and systems, starting with individual enterprises and organizations to regions, industries and the economy of the country in general. Griliches[6], Jorgenson[10], Murthy[11], Srivastava, and Gupta [13] studied the effects of various inputs on the production function. Mathematical models in industry have been extensively investigated by Hall [8], Anderson [1,2] and Andrews [3]. Brailsford, Potts, and Smith[5] introduce CSPs and their applications in operational research, such as scheduling and timetabling, which can be directly related to production functions. It provides a comprehensive overview of algorithms used to solve constraint satisfaction problem and their practical applications in different domains. Barták, Miguel and Salido[4] discuss the use of CSPs in planning and scheduling, highlighting how these methods can be applied to optimize production functions by ensuring that all constraints are met. It explores various techniques and methodologies that can be utilized to address these problems effectively. Barták and Salido present approaches to handle dynamic changes in CSPs, which is often the case in production environments where conditions and constraints can frequently change. Pandey and Gupta[12] explain the product development prototyping problem described in terms of the problem of limited satisfaction. The proposed method uses parametric estimates to identify relationships between different variables and restrictive programming to find variants to complete the project within the resources and project requirements of the company. Gupta [7] studied the Cobb-Douglas production function in general form and also explained the production behavior, profit maximization and cost structure of any industry. The aim of this paper is to develop and apply deterministic simulation models to analyze and optimize production within an industrial setting. It aims to contribute to a deeper understanding of production dynamics and also develop and implement deterministic simulation models to thoroughly analyze and optimize production functions within an industrial context while addressing constraint satisfaction.

2. Generalized Production Function

Analysis of production functions is often used as research because researchers need information about how to manage limited resources such as land, labor, and capital so that maximum production results are obtained. The initial model of the Cobb-Douglas production function.

The most popular production functions for two inputs are given below:

- Leontief production Function $Q(x_{1,} x_2)$= mini $(a_1 x_1 + a_2 x_2)$
- Linear Production Function $Q(x_{1,} x_2) = a_1 x_1 + a_2 x_2$
- Cobb Douglas Production Function $Q(x_{1,} x_2) = a_1 x_1^{a_1} a_2 x_2^{a_2}$

The Cobb-Douglas production with n inputs (such as capital, labor, land, raw materials, energy, technology, fertilizer, equipment etc.), is also called the generalized Cobb-Douglas production function.

A production function is a map Q of class. C^∞

$$Q : R^n \to R$$
$$Q = Q(x_i, x_2, x_3 \ldots \ldots . x_n)$$
$$= a_0 \prod_{i=1}^{n} x_i^{a_i}. \quad (1)$$

where $a_0 > 0$ and $a_i \epsilon (0,1)$ for every i = 1,2,3,……,n

The importance and advantage of the Cobb-Douglas production in its generalized form were very well outlined in [9] where Q is the quantity of output and $x_1, x_2, x_3 \ldots \ldots . x_n$ are input function. We recall now some of them with appropriate economic interpretation:

(i) $Q(0,0, \ldots \ldots .0) = 0$

(ii) $\frac{\partial Q}{\partial x_i} > 0, \forall\, i = 1,2,3 \ldots \ldots \ldots . n$

(iii) $\frac{\partial^2 Q}{\partial x_i^2} > 0, \forall i = 1,2,3 \ldots \ldots \ldots n$

(iv) (iv)$Q(x + y) \geq Q(x) + Q(y) >, \quad \forall\, x, y \in R^n$

(v) $Q(\lambda x + (1 - \lambda)y) \geq min\left(Q(x), Q(y)\right), \; \forall \lambda \in [0,1], \; x, y \in R^n$

We will denote by x=(x_1, x_2,.. x_i...x_n) in R^n vector of inputs, where $x_i \geq 0$ for i =1,2,3…..n by $(y_1, y_2, \ldots y_{i..} . y_n)$in R^n vector of product where $y_i \geq 0$ for $i = 1, \ldots, n, \; n \in N$, by the production process, we understand such a set of activities as a result of which a given bundle of inputs is transformed into a specific bundle of products. The production process is described by a nonnegative, 2n-dimensional vector. $(x, y) \geq 0$, in which x is the n-dimensional vector of inputs needed to produce the n-dimensional vector of products y,$n \in$ N vectors x and y form a technologically acceptable production process . A set $Z \in$ R^{2n} of all technologically acceptable production processes defined by the formula: $\|z\| = \max\{x_1, x_2, \ldots x_n, y_1, y_2 \ldots x_n\}$ For z ϵZ

We call it Z-production spaceThe production process (x, y) $\in Z$ we call technologically effective, if there is not exist another production process (x, y') $\in Z$ such that $y' > y$. With the technologically effective production process, a vector-valued production function is associated.

We define it in the following way

f:$R^n \rightarrow R$, such that $y=f(x)$, where x is a vector–valued function

If and only if the production process (x, y) ∈Z is technologically effective, then the function f is called vector–valued production function associated with p–production space Z. We will be concerned only with the case of the production process in which the manufacturer produces only one product using to this n-inputs, n ≤ N. In such a situation vector-valued production function defined above reduces to the inner, n - arguments production function f: $R^n \rightarrow R$, n≤ N, which is associated with the Z – production space $Z \subset R^{n+1}$. It is a function which maps each nonnegative vector of inputs x= $(x_1, x_2, \ldots, x_n)$ to such result of production $y=f(x)$, that the pair (x,y) makes a technologically effective production process. If a company operates in the long run (long–term strategy), let $x = (x_1 . x_2 \ldots . . x_n)$ be in puts vector of productive factors, where each productive factor satisfies the condition $x_i \geq 0$ for i = 1,2…n. Let further w = $(w_1, w_2, \ldots \ldots w_n,)$ be cost vector of productive factors. We will denote by ⟨w, x⟩ the inner product of vectors x and w:

$<w, x> = \sum_{i=1}^{k} w_i x_i$Let us consider the function f:$R^n \rightarrow R$ satisfying the condition (i)-(v). We will denote by $f(x)$ the quantity of fabricated product and p –the price of this product. We define profit function g: $R^n \rightarrow R$ of the company by the formula:

$$g(x) = pf(x) - \langle w, x \rangle \quad (2)$$

Where $pf(x)$ is the product sale income and ⟨w, x⟩ denotes the cost of the manufactured product. Our purpose is to maximize the company's profit when the production process is described by inner, n - arguments production function f: $R^n \rightarrow R$ determined in the definition. We are looking for such a vector of inputs of production factors $x^* \geq 0$, for which profit function g of the company, given by formula (2), riches its maximum.

3. Production Elasticity

The production elasticity shows the ratio of the relative change in output produced to the relative change in the number of inputs used.

The marginal productivity of $x_1, x_2, \ldots \ldots x_n$ are

$$(MP)_{x_i} \equiv \frac{\partial Q}{\partial x_i} = a_i \frac{Q}{x_i} \forall i = 1,2 \ldots . n$$

The average productivity relative to $x_1, x_2, \ldots \ldots x_n$ be

$$(AP)_{x_i} = \frac{Q}{x_i} \forall i = 1,2 \ldots . . n$$

The output elasticity of x_1 is denoted by e_{x_1} is measured through

$$e_{x_1} = \frac{(MP)_{x_1}}{(AP)_{x_1}} = \frac{\frac{\partial Q}{\partial x_1}}{\frac{Q}{x_1}} = a_1$$

Similarly $e_{x_i =}\ a_i \qquad \forall i = 1,2 \ldots n$

It can be seen that the output elasticity of the inputs can be measured directly through the coefficient of the production function.

4. Long-run analysis of the production function

Let us consider that initially the level of production function was Q_0 which is given as

$$Q_0 = a_0 x_1^{a_1} x_2^{a_2} x_3^{a_3} \ldots \ldots \ldots . x_n^{a_n}$$

Increasing all the input factors in the same proportion $\lambda > 0$, we get

$$Q^* = \lambda^{a_1 + a_2 + a_3 + a_4 + \cdots a_n} Q$$

$$a_1 + a_2 + a_3 + a_4 + \cdots a_n = \alpha$$

$$Q^* = \alpha Q$$

This shows that the production function is homogeneous this is measure of return to scale.

$\alpha = 1$, Constant return to scale

$\alpha < 1$, Decreasing return to scale

$\alpha > 1$, Increasing return to all

5. Deterministic Simulation Model for Generalised Production Function

A deterministic simulation model for a generalized Cobb-Douglas production function can help in understanding how different factors contribute to output under certain assumptions

$$Q = a_0 \prod_{i=1}^{n} x_i^{a_i}$$

$$lnQ \ = \ ln\, a_{\,0} + \ a_1 ln\, x_1 + \ a_2 ln\, x_2 \ + \ a_3 ln\, x_3 \ldots \ldots \ldots \ldots \ldots + a_n ln x_n$$

$$ln\left(\frac{w_1}{pa_1}\right) = ln\ Q - ln\ x_1$$

$$ln\left(\frac{w_2}{pa_2}\right) = ln\ Q - ln\ x_2$$

$$ln\left(\frac{w_n}{pa_n}\right) = ln\ Q - ln\ x_n$$

These equations may be written as

$$lnQ - a_1 ln\ x_1 - a_2 ln\ x_2 - a_3 ln\ x_3 \ldots\ldots\ldots\ldots\ldots - a_n lnx_n = \ln a_0$$

$$ln\ Q - ln\ x_1 = ln\left(\frac{w_1}{pa_1}\right)$$

$$ln\ Q - ln\ x_2 = ln\left(\frac{w_2}{pa_2}\right)$$

$$ln\ Q - ln\ x_n = ln\left(\frac{w_n}{pa_n}\right)$$

$$\begin{bmatrix} 1 & -a_1 & -a_2 & \ldots\ldots\ldots\ldots & -a_n \\ 1 & -1 & 0 & 0 & \ldots\ldots 0 \\ 1 & 0 & -1 & 0 & \ldots\ldots 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 0 & 0 & 0 & \ldots\ldots -1 \end{bmatrix} \begin{bmatrix} lnQ \\ lnx_1 \\ lnx_2 \\ lnx_3 \\ \ldots \\ lnx_n \end{bmatrix} = \begin{bmatrix} lna_0 \\ ln\frac{w_1}{pa_1} \\ ln\frac{w_2}{pa_2} \\ \ldots \\ ln\frac{w_n}{pn} \end{bmatrix}$$

$$\begin{bmatrix} lnQ \\ lnx_1 \\ lnx_2 \\ lnx_3 \\ \ldots \\ lnx_n \end{bmatrix} = \begin{bmatrix} 1 & \frac{a_1}{1+a_1} & \frac{a_2}{1+a_2} \ldots\ldots & \frac{a_n}{1+a_n} \\ 1 & 1+a_1 & a_1 \ldots\ldots\ldots\ldots & a_1 \\ 1 & a_2 & 1+a_2 \ldots\ldots\ldots & a_2 \\ & \ldots\ldots\ldots & \ldots\ldots\ldots & \\ 1 & a_n & a_n \ldots\ldots\ldots & 1+a_n \end{bmatrix} \begin{bmatrix} lna_0 \\ ln\frac{w_1}{pa_1} \\ ln\frac{w_2}{pa_2} \\ \ldots \\ ln\frac{w_n}{pn} \end{bmatrix}$$

Given that various companies have varying outputs and inputs, the deterministic model may prove overly limiting in its estimation, particularly when faced with identical price sets.

6. Optimization Conditions of an Industry

A firm is an competitive situation if it can buy and sell quanties at exogenously given prices ,which are independent of its production decision .The firm behaves so as to maximize the profit; by equation (2) $g(x)=pf(x)-\langle w, x\rangle$.

i.e the problem of the firm can be:

$$max\ g(x) = pf(x) - \langle w, x\rangle, \ldots\ldots \tag{3}$$

s.t .$x_i \geq 0$, i=1,2,3……n

Also the firm is trying to minimize its cost Z of the inputs used to produced output Q.

There the mathematical programming of expenditure of the firm is

$$mini\ Z = Z(x)$$

s.t. f(x) = Q (4)

and $x_i \geq 0$,i=1,2,3……n

closely interrelated to formulation (3) and (4) the problem of the firm with given level of expenditure or p respecifited budget Z ,and an objective function that maximizes the total output Q .Thus the firm chooses levels of inputs so as to maximize output . The mathematical formulation is given below

max Q = f(x)…… (5)

Subject to constraints $w_1x_1+w_2x_2+w_3x_3+\dots\dots\dots+w_n x_n \leq Z$

and $x_i \geq 0$ i=1,2,3……n

From equation (4) and (5) we formulate the generalized Cobb – Douglas production function:

$Q = a_0x_1^{a_1}x_2^{a_2}x_3^{a_3} \dots\dots\dots x_n^{a_n} = a_0\prod_{i=1}^{n} x_i^{a_i}$

where $a_i \in (0,1)$ for every i = 1,2,3,……,n

Hence we formulate the following constraint satisfaction problem (NLPP):

$mini\ Z = w_1x_1+w_2x_2+w_3x_3+\dots\dots\dots+w_nx_n$

Subject to constraints

$a_0\prod_{i=1}^{n} x_i^{a_i} \geq Q$ ……… (6)

and non-negative restrictions

$x_i \geq 0$,i=1,2,3……n

It is to find, for a particular output level Q and with a given structure of inputs prices, what input levels would constitute the cheapest way of producing this output and what would be minimum cost. This questions can be answered for all possible levels of output and the minimum cost would be depend on the level of output to be produced .

The optimal solution of problem (6) may be find by Kuhn-Tucker Conditions:

$$mini\ Z = \sum_{i=1}^{n} w_i x_i$$

Subject to

$$a_0 \prod_{i=1}^{n} x_i^{a_i} \geq Q$$

We introduced the Lagrange multiplier $\lambda \geq 0$ for the constraint. The Lagrangian function L is:

$$L(x_1, x_2 \dots\dots x_n) = \sum_{i=1}^{n} w_i x_i + \lambda(a_0 a_i \prod_{i=1}^{n} x_i^{a_i} - Q)$$

The KKT conditions require that for an optimal solution $(x_1, x_2 \dots\dots x_n)$

$$\frac{\partial L}{\partial x_i} = w_i + \lambda \frac{\partial}{\partial x_i}\left(a_0 \prod_{i=1}^{n} x_i^{a_i} - Q\right) = 0\ \ \forall i = 1,2,3 \dots\dots n$$

$$w_i + \lambda\left(a_0 a_i \prod_{i=1}^{n} x_i^{a_i} - Q\right) = 0$$

Simplifying we get

$$w_i + \lambda a_0 a_i \frac{P(x_1, x_2, \dots\dots x_n)}{x_i} = 0\ \ \forall i = 1,2,3, \dots . n$$

Stationary Condition:

$$w_i + \lambda a_0 a_i \frac{P(x_1, x_2, \ldots \ldots x_n)}{x_i} = 0$$

$\lambda = -\frac{w_i x_i}{\lambda a_0 a_i P(x_1, x_2, \ldots\ldots x_n)}$ $\forall i = 1,2,3 \ldots\ldots n$ because λ must be the same for all i=1,2,3…n , we can write

$$-\frac{w_1 x_1}{\lambda a_0 a_i P(x_1, x_2, \ldots\ldots x_n)} = -\frac{w_2 x_2}{\lambda a_0 a_i P(x_1, x_2, \ldots\ldots x_n)} = \cdots = -\frac{w_n x_n}{\lambda a_0 a_i P(x_1, x_2, \ldots\ldots x_n)}$$

Simplifying we get

$$-\frac{w_1 x_1}{a_1} = -\frac{w_2 x_2}{a_2} = \cdots = -\frac{w_n x_n}{a_n} = \xi(say)$$

we get

$$x_1 = -\frac{\xi a_1}{w_1}, x_2 = -\frac{\xi a_2}{w_2}, \ldots \ldots, x_n = -\frac{\xi a_n}{w_n}$$

7. Constraint Satisfaction for Generalised Production Function

The constraint typically considered for the Cobb-Douglas production function involve ensuring that the sum of elasticities equal 1 for constant return returns to scale (CRS), or less than 1 for decreasing return to scale (DRS) or greater than 1 for increasing return to scale (IRS).here are the possible constraints

1. Constant return to scale (CRS): $a_1 + a_2 + a_3 + a_4 + \cdots + a_n = 1$
2. Decreasing return to scale (DRS): $a_1 + a_2 + a_3 + a_4 + \cdots + a_n < 1$
3. Increasing return to scale(IRS) : $a_1 + a_2 + a_3 + a_4 + \cdots + a_n > 1$

Some additional constraint is also available

1. Non-negativity of Elasticities: $a_1, a_2, a_3, a_4, \ldots a_n \geq 1$
2. Non- negativity of Inputs: $x_1, x_2, \ldots \ldots x_n \geq 0$
3. Input Resource Limits: if there is maximum available quantities for each input , those would serve as upper bound

   $x_1 \leq x_{1_{max}}, x_2 \leq x_{2_{max}}, x_3 \leq x_{3_{max}}, \ldots \ldots \ldots, x_n \leq x_{n_{max}},$

8. Result and Discussion

A firm's objective is profit maximization. By analysing and abstracting the production function, this helps us to understand how distinct inputs (such as labour, capital, raw materials etc.) add value to the process of production. In other words, we have different types of input-output relationships that we can model by using various forms of production functions. Deterministic simulation models of the production function are used to estimate output given particular combinations of inputs. This involves writing down and solving equations that characterize how goods and services are produced. Optimizing a production function involves identifying the optimal combination of factors in order to either maximize output or minimize costs (subject to certain constraints such as a budget constraint). This study helps managers to make effective decisions about their operations through better management practices. Thus, they provide valuable insights on efficient decision-making within manufacturing setups leading to enhanced efficiency and increased productivity.

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