J. Nat. Acad. Math ISSN: 0970 5228
Vol. 36 (2022), pp. 40-48

# Curvature Tensors On Lorentzian $\beta$-Kenmotsu Manifold

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**Abstract**

The object of the present paper is to study some properties of Lorentgian $\beta$-Kenmotsu manifold. An irrotational pseudo projective curvature tensor in Lorentgian $\beta$-Kenmotsu manifold are studied. Finally we investigate pseudo $W_8$ flat Lorentgian $\beta$-Kenmotsu manifold satisfying the relation $R(X,Y).S = 0$, where $S$ is the Ricci tensor.

**Mathematics Subject Classification**. 53C25
**Keywords and phrases:** Lorentzian $\beta$-Kenmotsu Manifold, $\xi$-pseudo projective curvature tensor, irrotational curvature and pseudo $W_8$ curvature tensor $\widetilde{W}_8$ .

## 1 Introduction:

In 2002, Prasad [9] introduced and studied pseudo projective curvature tensor $\widetilde{P}$ [9] curvature tensor on a Riemannian manifold $M^n$ (n>2) by the following expression

$$\begin{aligned}\widetilde{P}(X,Y)Z =& aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] - \\ & \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{1.1}$$

where $a$ and $b$ are constant such that $a, b \neq 0$ and $r$ is the scalar curvature of the manifold. The pseudo projective curvature tensor have been studied by many authors such as Bagewadi, Prakash and Venkatesha [4], Narain, Prakash and Prasad [8], Jaisawal and Ojha [6], Gular and Demirabag [5], Kumar [7] etc. explored various geometrical properties by using this curvature tensor on LP-Sasakian manifold, K-contact manifold and trans-Sasakian manifold, weakly symmetric Riemannian manifold.

Recently, in 2018, Prasad, Yadav and Pandey [10] introduced and studied pseudo $W_8$ curvature tensor $\widetilde{W}_8$ on a Riemannian manifold $(M^n, g)$ by the following expression

Received: 28/03/2022; Accepted: 24/07/2022

$$\begin{aligned}\widetilde{W}_8(X,Y)Z =& aR(X,Y)Z + b[S(X,Y)Z - S(Y,Z)X] - \\ & \frac{r}{n}\left(\frac{a}{n-1} - b\right)[g(X,Y)Z - g(Y,Z)X],\end{aligned} \tag{1.2}$$

$\forall X, Y, Z \in TM$, where if $a = 1, b = \frac{1}{n-1}$ , then the pseudo $W_8$ curvature tensor $\widetilde{W}_8$ reduces to $W_8$ curvature tensor Pokhariyal [12].

# 2 Preliminaries:

A differentiable manifold of dimension n is called Lorentzian $\beta$-Kenmotsu manifold if it admits a (1,1)- tensor field $\phi$, a contravariant vector field $\xi$, a covariant vector field$\eta$ and Lorentzian metric g which satisfy ( Bagewadi and Venkatesha [1], Bagewadi and Kumar [3], [11])

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.1}$$

$$\phi^2 X = X + \eta(X)\xi, \quad g(X,\xi) = \eta(X), \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

for all X,Y $\in$ TM.

Also an Lorentzian $\beta$-Kenmotsu manifold M is satisfying

$$\nabla_X \xi = \beta[X - \eta(X)\xi], \tag{2.4}$$

$$(\nabla_X \eta)(Y) = \beta[g(X,Y) - \eta(X)\eta(Y)], \tag{2.5}$$

where $\nabla$ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Further, on an Lorentzian $\beta$-Kenmotsu manifold M the following relation hold: ( Bagewadi and Venkatesha [1], Bagewadi and Kumar [3])

$$\eta(R(X,Y)Z) = \beta^2[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)], \tag{2.6}$$

$$R(\xi.X)Y = \beta^2[\eta(Y)X - g(X,Y)\xi], \tag{2.7}$$

$$R(X,Y)\xi = \beta^2[\eta(X) - \eta(Y)X], \tag{2.8}$$

$$S(X,\xi) = -(n-1)\beta^2\eta(X), \tag{2.9}$$

$$Q\xi = -(n-1)\beta^2\xi, \tag{2.10}$$

$$S(\xi,\xi) = (n-1)\beta^2, \tag{2.11}$$

where S is the Ricci tensor of type (0,2) with Q is the Ricci operator i.e.

$$g(QX,Y) = S(X,Y). \tag{2.12}$$

Thew above result will be used in the next section.

# 3 $\xi$-pseudo projectively flat Lorentzian $\beta$-Kenmotsu:

**Definition3.1** The Lorentzian $\beta$-Kenmotsu manifold is called $\xi$-pseudo projectively flat if

$$\widetilde{P}(X,Y)\xi = 0. \tag{3.1}$$

Putting $\xi$ for Z in (1.1), we get

$$\begin{aligned}\widetilde{P}(X,Y)\xi =& aR(X,Y)\xi + b[S(X,\xi)Y - S(Y,\xi)X] - \\ & \frac{r}{n}\left(\frac{a}{n-1}+b\right)[g(X,\xi)Y - g(Y,\xi)X].\end{aligned} \tag{3.2}$$

Using (2.2), (2.8) and (2.9) in (3.2) we get

$$\begin{aligned}\widetilde{P}(X,Y)\xi = \left[\beta^2 + \frac{r}{n(n-1)}\right][a+(n-1)b] \\ [\eta(X)Y - \eta(Y)X].\end{aligned} \tag{3.3}$$

From (3.1) and (3.3), we get

$$r = -\beta^2 n(n-1), \qquad \textit{provided} \;\; a+(n-1)b \neq 0.$$

Hence we can state the following theorem:

**Theorem 3.1** *The scalar curvature tensor of a $\xi$-pseudo projectively flat Lorentzian $\beta$-Kenmotsu manifold $M^n$ is a constant, given by r=-$\beta^2 n(n-1)$ provided a+(n-1)b≠0.*

# 4 Irrotational curvature tensor in Lorentzian $\beta$-Kenmotsu manifold:

Irrotational curvature tensor R on a Riemannian manifold is given by (Bagewadi and Gatti [2])

$$\begin{aligned}Rot\widetilde{R} =& (\nabla_U R)(X,Y)Z + (\nabla_X R)(U,Y)Z + \\ & (\nabla_Y R)(U,X)Z - (\nabla_Z R)(X,Y)U).\end{aligned} \tag{4.1}$$

By virue of second Biachi identity, we have

$$(\nabla_U R)(X,Y)Z + (\nabla_X R)(U,Y)Z + (\nabla_Y R)(U,X)Z = 0.$$

Therefore (4.1) reduces to

$$Curl R = -(\nabla_Z R)(X,Y)U. \tag{4.2}$$

If the curvature tensor is irrotational then Curl R=0 and by (4.2) we have

$$(\nabla_Z R)(X,Y)U = 0, \tag{4.3}$$

which implies

$$\nabla_Z(R(X,Y)U) = R(\nabla_Z X, Y)U + R(X, \nabla_Z Y)U + R(X,Y)\nabla_Z U. \tag{4.4}$$

Putting $\xi$ for U in (4.4) and using (2.4) and (2.8), we have

$$R(X,Y)Z = -\beta^2[g(Y,Z)X - g(X,Z)Y]. \tag{4.5}$$

Contracting (4.5) with respect to X, we have

$$S(Y,Z) = -\beta^2(n-1)g(Y,Z), \tag{4.6}$$

which yields

$$r = -\beta^2 n(n-1).$$

Thus, we have the following theorem:

**Theorem 4.1** *If the curvature tensor in Lorentzian β-Kenmotsu manifold is irrotational then the manifold is Einstein and r=-$\beta^2 n(n-1)$.*

# 5 Irrotational pseudo projective curvature tensor in Lorentzian β-Kenmotsu manifold:

The relation of pseudo projective curvature tensor $\widetilde{P}$ on a Riemannian manifold is given by

$$\begin{aligned} Rot\widetilde{P} = &(\nabla_U \widetilde{P})(X,Y)Z + (\nabla_X \widetilde{P})(U,Y)Z + \\ &(\nabla_Y \widetilde{P})(U,X)Z - (\nabla_Z \widetilde{P})(X,Y)U. \end{aligned} \tag{5.1}$$

By view of second Biachi identity, we have

$$(\nabla_U \widetilde{P})(X,Y)Z + (\nabla_X \widetilde{P})(U,Y)Z + (\nabla_Y \widetilde{P})(U,X)Z = 0.$$

Therefore (5.1) reduces to

$$Curl\widetilde{P} = -(\nabla_Z \widetilde{P})(X,Y)U. \tag{5.2}$$

If the pseudo projective curvature tensor is irrotational then Curl$\widetilde{P}$=0 and by (5.2) we have

$$(\nabla_Z \widetilde{P})(X,Y)U = 0. \tag{5.3}$$

Thus, the manifold is pseudo projectively symmetric which implies that

$$R(X,Y)\widetilde{P} = 0. \tag{5.4}$$

From (5.4), it follows that

$$\begin{aligned} &g(R(X,Y)\widetilde{P}(U,V)W,\xi) - g(\widetilde{P}(R(X,Y)U,V)W,\xi) - \\ &g(\widetilde{P}(U,R(X,Y)V)W,\xi) - g(\widetilde{P}(U,V)R(X,Y)W,\xi) = 0. \end{aligned} \tag{5.5}$$

From (1.1), it can be easily seen that

$$\eta(\widetilde{P}(X,Y)\xi) = 0. \tag{5.6}$$

Putting $\xi$ for Y in (5.5), we obtain by virtue of (2.7) and (5.6), that is

$$\begin{aligned} &\beta^2[\widetilde{'P}(U,V,W,X) + \eta(X)\eta(\widetilde{P}(U,V)W) + g(X,U)\eta(\widetilde{P}(\xi,V)W) - \\ &\eta(\widetilde{P}(X,V)W)\eta(U) + \eta(\widetilde{P}(U,\xi)W)g(X,V) - \eta(\widetilde{P}(U,X)W)\eta(V) \\ &-\eta(W)\eta(\widetilde{P}(U,V)X)] = 0, \end{aligned}$$

$$where \quad \widetilde{'P}(U,V,W,X) = g(\widetilde{P}(U,V)W,X).$$

But $\beta^2 \neq 0$ then

$$\begin{aligned} &\widetilde{'P}(U,V,W,X) + \eta(X)\eta(\widetilde{P}(U,V)W) + g(X,U)\eta(\widetilde{P}(\xi,V)W) - \\ &\eta(\widetilde{P}(X,V)W)\eta(U) + \eta(\widetilde{P}(U,\xi)W)g(X,V) - \eta(\widetilde{P}(U,X)W)\eta(V) \\ &-\eta(W)\eta(\widetilde{P}(U,V)X)] = 0. \end{aligned} \tag{5.7}$$

Now, putting $U = X = e_i$ in (5.7) and taking summation over $1 \leq i \leq n$, we get

$$\begin{aligned} &\sum_{i=1}^{n} \widetilde{'P}(e_i,V,W,e_i) + (n-1)\eta(\widetilde{P}(\xi,V)W) - \eta(W)\sum_{i=1}^{n}\eta(\widetilde{P}(e_i,V)e_i) \\ &= 0. \end{aligned} \tag{5.8}$$

From it follows that

$$\sum_{i=1}^{n} \widetilde{'P}(e_i,V,W,e_i) = [a+b(n-1)]\left[S(V,W) - \frac{r}{n}g(V,W)\right], \tag{5.9}$$

$$\begin{aligned} \eta(\widetilde{P}(\xi,V)W) = &-bS(V,W) + \left[a\beta^2 + \frac{a+b(n-1)}{n(n-1)}\right]g(V,W) \\ &+ [a+b(n-1)]\left[\beta^2 + \frac{r}{n(n-1)}\right]\eta(V)\eta(W). \end{aligned} \tag{5.10}$$

$$\sum_{i=1}^{n} \eta(\widetilde{P}(e_i, V)e_i)\eta(W) = (a-b)\left[\frac{r}{n} + \beta^2(n-1)\right]\eta(V)\eta(W). \tag{5.11}$$

Using (5.9), (5.10) and (5.11) in (5.8), we obtain

$$aS(V,W) + a\beta^2(n-1)g(V,W) + b[r + n(n-1)\beta^2]\eta(V)\eta(W) = 0. \tag{5.12}$$

Putting V=W=$e_i$ in (5.12) and taking summation for $1 \leq i \leq n$, we get

$$(a-b)[r + n(n-1)\beta^2] = 0$$

$$r = -n(n-1)\beta^2, \quad a - b \neq 0. \tag{5.13}$$

by virtue of (5.13), we obtain (5.12) that is

$$S(V,W) = -(n-1)\beta^2 g(V,W), \quad a \neq 0. \tag{5.14}$$

In view of (2.6), (2.7), (5.13) and (5.14), it can be easily seen from (1.1) that is

$$\eta(\widetilde{P}(X,Y)Z) = 0, \quad if\ for\ all\ X,Y,Z. \tag{5.15}$$

Hence we can state the following theorem:

**Theorem 5.1** *If in Lorentzian β-Kenmotsu manifold the pseudo projective curvature tensor is irrotational then the manifold is pseudo projectively flat provided that a-b≠0.*

# 6 $\widetilde{W_8}$ Flat Lorentzian β-Kenmotsu manifold:

First we consider that the manifold is $\widetilde{W_8}$ flat. Hence from (1.2), we obtain

$$\begin{aligned} aR(X,Y)Z = &- b[S(X,Y)Z - S(Y,Z)X] + \\ &\frac{r}{n}\left(\frac{a}{n-1} - b\right)[g(X,Y)Z - g(Y,Z)X]. \end{aligned} \tag{6.1}$$

A Lorentzian β-Kenmotsu manifold is said to be Ricci-semi symmetric if

$$R(X,Y).S = 0, \tag{6.2}$$

where $R(X,Y)Z$ is treated as a derivation of the tensor algebra for any tangent vectors $X, Y$ and $S$ the Ricci tensor [11].

Also (6.2) gives

$$S(R(X,Y)Z,W) + S(Z,R(X,Y)W) = 0. \tag{6.3}$$

Using (5.1) in (5.3), we have

$$\begin{aligned} &-\frac{2b}{a}[S(X,Y)S(Z,W)-S(Y,Z)S(X,W)]\\ &+\frac{r}{n}\left(\frac{a}{n-1}-b\right)[2g(X,Y)S(Z,W)\\ &-g(Y,Z)S(X,W)-g(Y,W)S(X,Z)]=0. \end{aligned} \tag{6.4}$$

Putting $Y = Z$ in (6.4), we have

$$\begin{aligned} &-\frac{2b}{a}[S(X,Z)S(Z,W)-S(Z,Z)S(X,W)]\\ &+\frac{r}{n}\left(\frac{a}{n-1}-b\right)[2g(X,Z)S(Z,W)\\ &-g(Z,Z)S(X,W)-g(Z,W)S(X,Z)]=0. \end{aligned} \tag{6.5}$$

Putting $\xi$ for Z and using (2.1), (2.2), (2.9) and (2.10), we have

$$\begin{aligned} &\left[\frac{2b(n-1)\beta^2}{a}+\frac{r}{n}\left(\frac{a}{n-1}-b\right)\right]S(X,W)-\\ &\left[\frac{2b(n-1)^2\beta^4}{a}+\frac{r}{n}\left(\frac{a}{n-1}-b\right)(n-1)\beta^2\right]\eta(X)\eta(W)=0. \end{aligned} \tag{6.6}$$

From eq$^n$ (2.12) and (6.6), we get

$$\begin{aligned} &\left[\frac{2b(n-1)\beta^2}{a}+\frac{r}{n}\left(\frac{a}{n-1}-b\right)\right]g(QX,W)-\\ &\left[\frac{2b(n-1)^2\beta^4}{a}+\frac{r}{n}\left(\frac{a}{n-1}-b\right)(n-1)\beta^2\right]\eta(X)\eta(W)=0. \end{aligned} \tag{6.7}$$

Let $\lambda$ be the eigen value of the endomorphism Q corresponding to an eigen vector X. Then

$$QX = \lambda X. \tag{6.8}$$

Using (6.8) in (6.7), then using (2.9), we have

$$\begin{aligned} &\left[\frac{2b(n-1)\beta^2}{a}+\frac{r}{n}\left(\frac{a}{n-1}-b\right)\right]\lambda g(X,W)-\\ &\left[\frac{2b(n-1)^2\beta^4}{a}+\frac{r}{n}\left(\frac{a}{n-1}-b\right)(n-1)\beta^2\right]\eta(X)\eta(W)=0. \end{aligned} \tag{6.9}$$

Putting $\xi$ for $W$ in (6.9) and using (2.1) and (2.2), we get

$$\begin{aligned} &\Bigg[\lambda\left\{\frac{2b(n-1)\beta^2}{a}+\frac{r}{n}\left(\frac{a}{n-1}-b\right)\right\}+\frac{2b(n-1)^2\beta^4}{a}+\\ &\frac{r}{n}\left(\frac{a}{n-1}-b\right)(n-1)\beta^2\Bigg]\eta(X)=0. \end{aligned}$$

As $\eta(X) \neq 0$, we have

$$\left[\lambda\left\{\frac{2b(n-1)\beta^2}{a} + \frac{r}{n}\left(\frac{a}{n-1} - b\right)\right\} + \frac{2b(n-1)^2\beta^4}{a} + \frac{r}{n}\left(\frac{a}{n-1} - b\right)(n-1)\beta^2\right] = 0. \tag{6.10}$$

From 6.10), we get one non-null solution

$$\lambda = -\frac{\frac{2b(n-1)^2\beta^4}{a} + \frac{r}{n}\left(\frac{a}{n-1} - b\right)(n-1)\beta^2}{\frac{2b(n-1)\beta^2}{a} + \frac{r}{n}\left(\frac{a}{n-1} - b\right)}. \tag{6.11}$$

Again from (5.1) we have

$$\begin{aligned} ag(R(X,Y)Z,W) = &- b[S(X,Y)g(Z,W) - S(Y,Z)g(X,W)] + \\ &\frac{r}{n}\left(\frac{a}{n-1} - b\right)[g(X,Y)g(Z,W) - g(Y,Z)g(X,W)]. \end{aligned} \tag{6.12}$$

Putting $X = W$, (5.10) reduses to

$$\begin{aligned} ag(R(W,Y)Z,W) = &- b[S(W,Y)g(Z,W) - S(Y,Z)g(W,W)] + \\ &\frac{r}{n}\left(\frac{a}{n-1} - b\right)[g(W,Y)g(Z,W) - g(Y,Z)g(W,W)]. \end{aligned} \tag{6.13}$$

The sum for $1 \leq i \leq n$ for the above expression for $W = e_i$ yields.

$$[a - (n-1)b]\left[S(Y,Z) + \frac{r}{n}g(Y,Z)\right] = 0, \tag{6.14}$$

Eq$^n$ (6.14) gives

$$\left[S(Y,Z) + \frac{r}{n}g(Y,Z)\right] = 0, \quad \textit{provided} \;\; a - (n-1)b \neq 0. \tag{6.15}$$

Putting $\xi$ for $Z$ in (6.15) and using (2.1), (2.2), (2.9) and (2.10), we have

$$r = \beta^2.n(n-1).$$

Hence, we can state the following theorem:

**Theorem 6.1** *In Lorentzian β-Kenmotsu manifold $M^n$ which is $\widetilde{W}_8$ flat together with $R(X,Y).S = 0$, the symmetric endomorphism $Q$ of tangent space corresponding to $S$ has only one non-zero eigen value given by (6.11) . Also, the scalar curvature of $\widetilde{W}_8$ flat is a constant, given by $r = \beta^2.n(n-1)$, provided $[a - (n-1)b] \neq 0$.*

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